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Formality related to universal enveloping algebras and study of Hom-(co)Poisson algebras

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General introduction

This thesis aims to study some algebraic aspects of structures linked to the problem of deformation quantization. At first, we examine the formality for the case of free algebras and for the Lie algebra $\mathfrak{so}(3)$ and then, we consider deformation quantization for Hom-algebraic structures. The following is about the historic of these subjects, the results are exposed with more details at the beginning of each part.

Deformation of structure theories is useful to formalize quantum physics. If one has a quantum description of a physical system, then the passage to the classical description is done letting "Planck's constant $\hbar$ tend to zero". The reverse operation, *i.e.* producing a quantum description from a classical one is called *quantization*. The algebraic structure considered by classical mechanics is the associative commutative algebra of smooth functions over a symplectic, or more generally, a Poisson manifold. Deformation quantization consists to construct an associative non-commutative multiplication (more precisely a $\star$-product) on the formal series in $\hbar$ with coefficients in this algebra, which encodes the Poisson bracket in the first order. The Poisson structure is then called the quasi-classical limit and the deformation is the $\star$-product. This point of view, initiated in 1978 in [BFF+78], tries to consider quantum mechanics as a deformation from classical mechanics, and the Poincaré group as a deformation of the Galileo group. This pioneering work raises the fundamental questions about the existence and the uniqueness of a deformation quantization for a given Poisson manifold.

The first results treated the case of symplectics manifolds. The general case of Poisson manifolds was solved by Kontsevich in 1997 in [Kon03]. He deduced the result by proving a much more general statement, which he called "formality conjecture". Endowed with the Gerstenhaber bracket, the Hochschild complex of the algebra of smooth functions over a Poisson manifold admits a graded Lie algebraic structure by shift, which controls the deformations of the Poisson bracket. Kontsevich shows that this complex is linked with its cohomology — which therefore controls the same deformations — by a L_∞-quasi-isomorphism, called a formality map. This boils down the problem of deformation to the previously solved case.

Konstevich shows in particular that the formality criterion is true for symmetric algebras over a finite-dimensional vector space are formal. Bordemann and Makhlouf have examined in [BM08] a slight generalization to the case of universal enveloping algebras. They showed (implicit in Kontsevich's work) that the formality for a Lie algebra is equivalent to those of its universal enveloping algebra. They also proved that there is formality for the universal enveloping algebra of an affine Lie algebra. These methods were used in [BMP05] and give informations about the rigidity of universal enveloping algebras.

In the first part, we will study the question of formality of some classes of Lie algebras. We consider free algebras, a particular case of universal enveloping algebras, and we show that there is no formality in general, except in the trivial cases. The study of the Lie algebra $\mathfrak{so}(3)$ shows that there is no formality in this case too.

The tools used are homological ones. We first recall that the cohomology is concentrated in degrees 0 and 1 for the free algebras and in degrees 0 and 3 for the Lie algebra $\mathfrak{so}(3)$. We then build a L_∞-quasi-isomorphism between the differential graded Lie algebra of Hochschild's cochains endowed with the Gerstenhaber bracket and the cohomology endowed with the Schouten bracket.

To achieve this, we use a version of the *Perturbation lemma* adapted to differential graded Lie algebras, which states that given a contraction between two differential graded complex, and a perturbation of one of the differentials, there exists a new contraction between the two complexes endowed with perturbed differentials.

The study of quasi-deformations of Lie algebra of vector fields, in particular the q-deformations of Witt and Virasoro algebra, leads to the introduction of new non-associative structures. Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in order to describe theses q-deformations using σ-derivations (see [HLS06]). They describe the q-deformation of the Witt algebra by a one-parameter q family of Hom-Lie algebras, such that the initial Witt algebra is obtained for $q = 1$. The associative type objects corresponding to Hom-Lie algebras, called Hom-associative algebras, were introduced by Makhlouf and Silvestrov in [MS10b]. The enveloping algebras of Hom-Lie algebras were studied by D. Yau in [Yau08]. The dual notions of Hom-coalgebras, and also Hom-bialgebras, Hom-Hopf algebras and Hom-Lie coalgebras were first studied in [SPAS09, MS10a] and have been enhanced in [Yau11, Yau10a].

The formal deformation theory is extended in [MS10b] to Hom-associatives and Hom-Lie algebras. The theory for bialgebras and Hopf algebras was introduced in [GS92], and was extended to the Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras in [DM]. Cohomology complexes were build for theses different algebraic structures, and links between cohomology and formal deformation established.

The problem of deformation quantization in the Hom-algebraic setting can be formulated as follows: for a given Hom-(co)Poisson algebra, it is to find a deformation of a commutative (co)algebra Hom-(co)associative such that the first term of the deformation corresponds to the initial Hom-(co)Poisson algebra.

The second part of this work was in large part prepublished in the article [BEM12]. In this second part, we recall some properties of Hom-algebraic structures, we introduce the notion of Hom-co-Poisson algebra and study duality. We use a deformation principle by twist to build new structures of the same type, or to deform a classical structure into the corresponding Hom-structure by means of an algebra morphism. In particular, we apply this process to Poisson structures and to $\star$-products of Moyal-Weyl.

We also establish a correspondence between universal enveloping algebras from Hom-Lie algebras endowed with a Hom-coPoisson structure and Hom-Lie bialgebras.

Part I

Formality and L_∞ structures

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Introduction of the first part

In this first part, after presenting graded structures in Section 1.1 and recalling cohomology notions in Section 1.2, we define the notions of L_∞-algebras and formality (Definition 2.1.2 and Definition 2.1.3). A L_∞-algebra is a generalization of a graded Lie algebra. It is some differential graded cocommutative symmetric coalgebra, whose differential encodes a Lie bracket and maps of greater arity. The fact that the differential is of square zero by composition encompass the Jacobi identity at order two and others identities at any order involving these maps.

A L_∞-morphism (Definition 2.1.2) is a map between two L_∞-algebras which intertwine the differentials. Considering the components of the morphism, this also gives equations at any order.

For an associative algebra, we consider its Hochschild complex and its cohomology which, endowed with the Gerstenhaber bracket and the induced Schouten bracket, become graded Lie algebras by shift. They can be considered as L_∞-algebras, differentials are induced by the brackets. In general, there is no graded Lie algebra morphism injecting the cohomology into the cocycles, but maybe there exists a morphism of L_∞-algebras intertwining the differentials induced by the brackets.

A formality map is such a morphism, if it induce an isomorphism on cohomology (a quasi-isomorphism). If a formality map does not exist, it is always possible to modify order by order the L_∞ structure of the cohomology: the differential is then no longer induced by the Schouten bracket only, but contains higher order components. This construction use a version of the *Perturbation lemma* (Lemma 2.5.4) adapted for differential graded Lie algebras.

We then study the formality equations for free algebras, and show in Section 3.2 that they are not satisfied. The computation of the perturbed L_∞ structure gives only one more map of arity 3. The same work is done for the Lie algebra $\mathfrak{so}(3)$. Again, there is no formality (Theorem 4.4.1) and the modified L_∞ structure consists of only one term of arity 3 too.

1

Preliminaries

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To begin, we expose the workplace, which is the category of graded vector spaces. The notion of components shift plays an important role an allows to enrich the involved structures. We then detail some of the structures which can equip tensor spaces. This chapter ends by reminding the notions of Hochschild and Chevalley-Eilenberg cohomology, and some of their properties

1.1 Graded structures

A gradation on a space enable the concise exposition of valid properties in each degree. Computations are also easier by working with homogeneous elements in each component. In the following, $\mathbb{K}$ is a field of characteristic 0, unless otherwise stated.

1.1.1 Graded vector spaces

We consider the *category of $\mathbb{Z}$-graded vector spaces*: objects are $\mathbb{K}$-vector spaces $\mathbb{Z}$-graded $V = \oplus_{i\in\mathbb{Z}} V^i$, direct sum of subspaces V^i. An element x of V lying in one of the V^i is called *homogeneous*, and we shall denote by $i =: |x| \in \mathbb{Z}$ its degree. In the following, elements will always be homogeneous unless otherwise specified. Given two graded vector spaces V and W, a linear map $\phi : V \to W$ is said to be homogeneous of degree j if and only if for all integers i we have $\phi(V^i) \subset W^{i+j}$. In graded situations we shall write $\mathrm{Hom}(V,W)^j$ for the vector space of all linear maps which are homogeneous of degree j, and $\mathrm{Hom}(V,W)$ for the direct sum of all the $\mathrm{Hom}(V,W)^j$. Clearly, $\mathrm{Hom}(V,W)$ is a graded vector space.

Likewise, the tensor product $V \otimes W$ is graded by setting $(V \otimes W)^i = \oplus_{k\in\mathbb{Z}} V^k \otimes W^{i-k}$, see [ML63]. The tensor product of two morphism $\phi : V \to W$ and $\psi : V' \to W'$ is defined with the *Koszul rule of signs*: for all homogeneous elements $x \in V$ and $y \in V'$

$$(\phi \otimes \psi)(x \otimes y) := (-1)^{|\psi||x|} \phi(x) \otimes \psi(y), \tag{1.1.1}$$

with ψ of degree $|\psi|$. There also is the *graded transposition*

$$\begin{aligned} \tau : V \otimes W &\to W \otimes V \\ \tau(x \otimes y) &:= (-1)^{|x||y|} y \otimes x. \end{aligned} \tag{1.1.2}$$

Theses two rules will determine all the signs which will appears in computations.

A graded (associative) algebra $(\mathcal{A}, \mu)$ is a graded vector space $\mathcal{A}$ together with graded space morphism $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ of degree 0 satisfying

$$\mathcal{A}^i \mathcal{A}^j \subset \mathcal{A}^{i+j}.$$

Very often we write aa' for $\mu(a \otimes a')$. For another graded algebra $(\mathcal{B}, \nu)$ the *graded tensor product of $\mathcal{A}$ and $\mathcal{B}$* is $\mathcal{A} \otimes \mathcal{B}$ with the product map defined as the composite $(\mu \otimes \nu) \circ (id \otimes \tau \otimes id)$. In terms of elements, the product is given for homogeneous elements $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ by

$$(a \otimes b)(a' \otimes b') := (-1)^{|b||a'|} aa' \otimes bb'.$$

A graded algebra $(\mathcal{A}, \mu)$ is *commutative* if $\mu \circ \tau = \mu$, *anticommutative* if $\mu \circ \tau = -\mu$; for $a, b \in \mathcal{A}$ this writes

$$ab = (-1)^{|a||b|} ba \qquad \text{et} \qquad ab = -(-1)^{|a||b|} ba.$$

A *graded coalgebra* $(\mathcal{C}, \Delta)$ is defined in a similar way: the comultiplication must satisfy

$$\Delta \mathcal{C}^j \subset \sum_{k+l=j} \mathcal{C}^k \otimes \mathcal{C}^l.$$

It is said *cocommutative* if $\tau \circ \Delta = \Delta$ and *anticocommutative* if $\tau \circ \Delta = -\Delta$. We define a graded coalgebra structure on the tensor product of two graded coalgebras by applying the same rule of signs as in the case for algebras.

A *derivation of degree* i in a graded algebra $(\mathcal{A}, \mu)$ is a linear morphism $d : \mathcal{A} \to \mathcal{A}$ of degree i such that

$$d(ab) = da.b + (-1)^{i|a|} a.db$$

which also writes

$$d \circ \mu = \mu \circ (d \otimes id_{\mathcal{A}} + id_{\mathcal{A}} \otimes d).$$

A *coderivation of degree* i in a graded coalgebra $(\mathcal{C}, \Delta)$ is a linear morphism $d : \mathcal{C} \to \mathcal{C}$ of degree i such that if we note $\Delta a = \sum_{(a)} a_1 \otimes a_2$, we have

$$\Delta da = \sum_{(a)} da_1 \otimes a_2 + (-1)^{i|a_1|} a_1 \otimes da_2$$

or also

$$\Delta \circ d = (d \otimes id_{\mathcal{C}} + id_{\mathcal{C}} \otimes d) \circ \Delta.$$

Definition 1.1.1 A *graded Lie algebra* is a $\mathbb{Z}$-graded vector space V endowed with a graded Lie bracket, *i.e.* a bilinear map $[\ ,\ ] : V \otimes V \to V$ such that

gradation $[V^i, V^j] \subset V^{i+j}$
and for $x, y, z \in V$

graded antisymmetry

$$[y, x] = -(-1)^{|y||x|} [x, y] \qquad (1.1.3)$$

graded Jacobi identity

$$(-1)^{|x||z|} [[x, y], z] + (-1)^{|y||x|} [[y, z], x] + (-1)^{|z||y|} [[z, x], y] = 0 \qquad (1.1.4)$$

which also writes

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|} [[x, y], z] = 0,$$

where $\circlearrowleft_{x,y,z}$ indicates a summation on cyclic permutations of x, y, z.

1.1.2 Shifted spaces

For an integer j denote by $V[j]$ the *shifted graded vector space* defined by $V[j]^i := V^{i+j}$. The identity map $V \to V$ induces for each $n \in \mathbb{Z}$ a map $s^n : V[j] \to V[j-n]$ which is of degree n because $s^n(V[j]^k) = V^{j+k} = V[j-n]^{k+n}$. It is seen as the n^{th} power of the *suspension map* $s := s^1 : V[j] \to V[j-1]$ of degree one.

In particular, $s : V[1] \to V =: V[0]$ and if an element $x \in V[1]$ is of degree $|x|$ (in $V[1]$), then $x = sx$ is of degree $|sx| = |x|+1$ in V.

The suspension will be 'visible' for *shifted multilinear maps*: let $\phi : V^{\otimes k} \to W^{\otimes l}$ be a multilinear map of degree $|\phi|$. The shifted map $\phi[j] : V[j]^{\otimes k} \to W[j]^{\otimes l}$ is defined by setting $\phi[j] := (s^{\otimes l})^{-j} \circ \phi \circ (s^{\otimes k})^j$. The degree of the shifted map $\phi[j]$ is given by $|\phi[j]| = j(k-l) + |\phi|$ and we have $(\phi[j])[j'] = \phi[j+j']$. Note that $(s^{\otimes k})^j = (-1)^{\frac{k(k-1}{2}\frac{j(j-1)}{2}}(s^j)^{\otimes k}$. In order to compute a shifted map, first write for $\xi := x_1 \otimes \cdots \otimes x_k \in V^{\otimes k}$ the value of $\phi(\xi)$ with the Sweedler notation as $\sum \phi_{(1)}(\xi)\otimes\cdots\otimes\phi_{(l)}(\xi)$, with $\phi_{(i)}(\xi) \in W$. By the Koszul rule of signs (1.1.1), the value of the shifted map $\phi[j]$ on $\eta := y_1 \otimes \cdots \otimes y_k \in V[j]^{\otimes k}$ is computed as follows, with $\tilde{\eta} := s^j(y_1)\otimes\cdots\otimes s^j(y_k)$:

$$\begin{aligned}
&\phi[j](y_1 \otimes\cdots\otimes y_k) = \\
&\quad (-1)^{\frac{k(k-1)}{2}\frac{j(j-1)}{2}+\frac{l(l-1)}{2}\frac{-j(-j-1)}{2}}(-1)^{j\left((k-1)|y_1|+(k-2)|y_2|+\cdots+(k-(k-1))|y_k|\right)} \\
&\quad \sum(-1)^{j\left((l-1)|\phi_{(1)}(\tilde{\eta})|+(l-2)|\phi_{(2)}(\tilde{\eta})|+\cdots+(l-(l-1))|\phi_{(l)}(\tilde{\eta})|\right)} \\
&\quad \phi_{(1)}(y_1\otimes\cdots\otimes y_k)\otimes\cdots\otimes\phi_{(l)}(y_1\otimes\cdots\otimes y_k).
\end{aligned}$$

1.1.3 Tensor bialgebra

For a $\mathbb{Z}$-graded vector space V, we denote by $\mathcal{T}V = \oplus_{k\in\mathbb{N}} V^{\otimes k}$ the tensorial algebra over V. It is a $\mathbb{K}$-associative graded algebra with unit $\mathbf{1}$, it inherits its $\mathbb{Z}$-grading by the $\mathbb{Z}$-grading of V, see [ML63]. To avoid any confusion, the symbol $\otimes$ is not written for the free multiplication $\mu = \mu_{\mathcal{T}V}$ in $\mathcal{T}V$, which is the juxtaposition.

Moreover, $\mathcal{T}V$ is a graded bialgebra: let $\mathcal{T}V^+ = \oplus_{k\in\mathbb{N}^\star} V^{\otimes k}$ be the augmentation ideal. The counit $\varepsilon = \varepsilon_{\mathcal{T}V} : \mathcal{T}V \to \mathbb{K}$ is defined by the condition $\operatorname{Ker}\varepsilon = \mathcal{T}V^+$ and $\varepsilon(\mathbf{1}) = 1_{\mathbb{K}}$. The graded *comultiplication shuffle* Δ_{sh} is the morphism of associative algebra $\mathcal{T}V \to \mathcal{T}V\otimes\mathcal{T}V$ induced (by universal property Theorem 1.1.2) by its value $\Delta_{sh}(x) = x\otimes\mathbf{1}+\mathbf{1}\otimes x$ on generators $x \in V$.

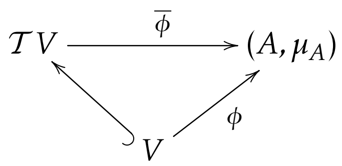
Since the multiplication $\mu^{[2]}$ on $\mathcal{T}V\otimes\mathcal{T}V$ is given by $(\mu\otimes\mu)\circ(id\otimes\tau\otimes id)$ with the graded transposition, there are signs in formulas involving Δ_{sh}, for example $\Delta_{sh}(xy) = xy\otimes\mathbf{1}+x\otimes y+(-1)^{|x||y|}y\otimes x+\mathbf{1}\otimes xy$, for $x \in V^{|x|}$ and $y \in V^{|y|}$. This comultiplication is graded cocommutative (*i.e.* $\tau\circ\Delta_{sh} = \Delta_{sh}$) of degree 0.

Dualizing this bialgebra structure, we obtain on the space $\mathcal{T}V$ another structure of bialgebra, with comultiplication (not graded cocommutative) of *deconcatenation*, $\Delta = \Delta_{\mathcal{T}V}$ which dualize the free multiplication and is given by formula $\Delta(x_1 \cdots x_k) = \mathbf{1} \otimes x_1 \cdots x_k + \sum_{r=2}^{k} x_1 \cdots x_{r-1} \otimes x_r \cdots x_k + x_1 \cdots x_k \otimes \mathbf{1}$; and *multiplication shuffle* μ_{sh} graded commutative, sometimes written $\bullet$, which dualize the comultiplication shuffle. For an explicit formula of μ_{sh}, see equation (1.1.9). The two operations Δ and μ_{sh} are of degree 0, the unit is again $\mathbf{1}$ and the counit ε. We note pr_V the canonical projection on $\mathcal{T}V^1$.

For two linear maps ψ_1, ψ_2 from a graded coassociative coalgebra (C, Δ_C) to a graded associative algebra (A, μ_A), the convolution $\psi_1 \star \psi_2$ of ψ_1 and ψ_2 with respect to μ_A and Δ_C is given by $\psi_1 \star \psi_2 := \mu_A \circ (\psi_1 \otimes \psi_2) \circ \Delta_C$. It is a graded associative multiplication in $\mathrm{Hom}(C, A)$.

The tensorial algebra $(\mathcal{T}V, \mu = \mu_{\mathcal{T}V}, \mathbf{1})$ is *free*[1] in the sense that it is characterized up to isomorphism by the following universal property.

Theorem 1.1.2 *Let (A, μ_A) be an associative graded algebra. Each morphsim $\phi : V \to A$ of graded vector spaces, of degree zero, extends to a unique morphism $\overline{\phi} : \mathcal{T}V \to A$ of graded algebras.*



$$\begin{aligned} &\overline{\phi} \quad \text{satisfies for each } n \in \mathbb{N} \\ &\overline{\phi}(x_1 \cdots x_n) = \phi(x_1) \cdots \phi(x_n) \\ \Leftrightarrow\ &\overline{\phi} \circ \mu^{(n-1)} = \mu_A^{(n-1)} \circ (\phi \otimes \cdots \otimes \phi) \end{aligned} \tag{1.1.5}$$

Proof. The proof given here presents a construction slightly more explicit than the usual one using the universal property of the tensorial product.
Let $\phi^{\star n} = \mu_A^{(n-1)} \circ \phi^{\otimes n} \circ \Delta^{(n-1)}$, with $\phi^{\star 0} = 1_A \varepsilon_{\mathcal{T}V}$ and $\phi^{\star 1} = \phi$, and let $\overline{\phi} = \sum_{n \in \mathbb{N}} \phi^{\star n}$. Extending ϕ by 0 on $\oplus_{n \neq 1} V^{\otimes n}$, we have $\phi^{\otimes k+1} \Delta^{(k)}(x_1 \cdots x_n) = 0$ for $k > n$ so $\overline{\phi}$ is well-defined. Thus, in the sum defining $\overline{\phi}(x_1 \cdots x_n)$, the only remaining term is exactly $\phi(x_1) \cdots \phi(x_n)$. So $\overline{\phi}$ is uniquely determined by ϕ. □

The algebra morphism $\overline{\phi}$ induced by $\phi : V \to A$ is computed as $\overline{\phi} = \sum_{n \in \mathbb{N}} \phi^{\star n}$, the geometric serie using the convolution with respect to the multiplication μ_A and the comultiplication of deconcatenation Δ.

Proposition 1.1.3 *Let $d : V \to A$ be a linear map of degree $j \in \mathbb{Z}$. There exists a unique graded derivation of degree j along $\overline{\phi}$, noted $\overline{d} : \mathcal{T}V \to A$, i.e. $\overline{d} \circ \mu_A =$*

[1] and so often called free algebra

$\mu_A \circ (\overline{d} \otimes \overline{\phi} + \overline{\phi} \otimes \overline{d})$, such that $\overline{d}|_V = d$.

Proof. Set $\overline{d}(x) := d(x)$ for $x \in V$, then extend using the formula $\overline{d}(xy) = \overline{d}(x)\overline{\phi}(y) + \overline{\phi}(x)\overline{d}(y)$. Since it respects the associativity of the free multiplication of $\mathcal{T}V$

$$\overline{d}(x(yz)) = d(x)\overline{\phi}(yz) + \phi(x)\overline{d}(yz) = d(x)\phi(y)\phi(z) + \phi(x)d(y)\phi(z) + \phi(x)\phi(y)d(z)$$
$$\overline{d}((xy)z) = \overline{d}(xy)\phi(z) + \overline{\phi}(xy)d(z) = d(x)\phi(y)\phi(z) + \phi(x)d(y)\phi(z) + \phi(x)\phi(y)d(z)$$

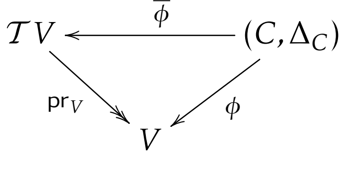
for all $x, y, z \in V$, the proposition follows. □

This derivation induced by d is computed as $\overline{\phi} \star d \star \overline{\phi}$. In term of elements, for $x_1 \cdots x_k \in \mathcal{T}V$, it writes

$$\overline{d}(x_1 \cdots x_k) = \sum_{r=1}^{k} \phi(x_1) \cdots \phi(x_{r-1}) d(x_r) \phi(x_{r+1}) \cdots \phi(x_k).$$

A graded coalgebra $(C, \Delta_C, \varepsilon_C, 1_C)$ is said to be *augmented* if C is the direct sum $C = \mathbb{K}1_C \oplus \mathrm{Ker}\,\varepsilon_C$. The subspace $C^+ :=$ Ker ε_C is isomorphic to the graded quotient coalgebra $C/\mathbb{K}1_C$ whitout counit ($\mathbb{K}1_C$ is a subcoalgebra, thus a coideal of C). A graded coalgebra whitout counit is said to be *nilpotent* if for each element $x \in C$, there is an integer N such that the N^{th} iteration of the comultiplication vanishes on x. Augmented graded coalgebras $(C, \Delta_C, \varepsilon_C, 1_C)$ whose C^+ (seen as a quotient) are nilpotent form a subcategory $\mathcal{C}_{AN}$ of the category of graded coalgebras. The category $\mathcal{C}_{AN}$ is closed under tensorial product and contains $\mathcal{T}V$. The coalgebra $(\mathcal{T}V, \Delta = \Delta_{\mathcal{T}V}, \varepsilon)$ is *cofree* in the category $\mathcal{C}_{AN}$ in the sense that it satisfies the following universal property.

Theorem 1.1.4 *For each coalgebra $(C, \Delta_C, \varepsilon_C) \in \mathcal{C}_{AN}$ and each linear map $\phi : C \to V$ of degree 0 vanishing on 1_C, there exists a unique morphism of graded coalgebras $\overline{\phi} : C \to \mathcal{T}V$ such that $\mathrm{pr}_V \circ \overline{\phi} = \phi$.*



$$\overline{\phi} \quad \textit{satisfies} \quad \Delta \circ \overline{\phi} = \overline{\phi} \otimes \overline{\phi} \circ \Delta_C \Leftrightarrow \sum_{(\overline{\phi}(x))} \overline{\phi}(x)_1 \otimes \overline{\phi}(x)_2 = \sum_{(x)} \overline{\phi}(x_1) \otimes \overline{\phi}(x_2) \tag{1.1.6}$$

Proof. Let $\phi^{\star n} = \mu^{(n-1)} \circ \phi^{\otimes n} \circ \Delta_C^{(n-1)}$, with $\phi^{\star 0} = \mathbf{1}\,\varepsilon_C$ and $\phi^{\star 1} = \phi$, and let $\overline{\phi} = \sum_{n \in \mathbb{N}} \phi^{\star n}$, it is well-defined by nilpotency. For $x \in \mathbb{K}1_C$, all equations vanish. For $x \in \mathrm{Ker}\,\varepsilon_C$, we have, in Sweedler notation

$$\begin{aligned}
\Delta\circ\overline{\phi}(x) =& \mathbf{1}\otimes\phi(x)+\phi(x)\otimes\mathbf{1}\\
&+\mathbf{1}\otimes\phi(x_1)\phi(x_2)+\phi(x_1)\otimes\phi(x_2)+\phi(x_1)\phi(x_2)\otimes\mathbf{1}\\
&+\Delta(\phi(x_1)\phi(x_2)\phi(x_3))+\cdots\\
(\overline{\phi}\otimes\overline{\phi})\circ\Delta_C(x) =& \mathbf{1}\,\varepsilon_C(x_1)\otimes\phi(x_2)+\phi(x_1)\otimes\mathbf{1}\,\varepsilon_C(x_2)\\
&+\mathbf{1}\,\varepsilon_C(\phi(x_1))\otimes\phi(x_2)\phi(x_3)\\
&+\phi(x_1)\otimes\phi(x_2)\\
&+\phi(x_1)\phi(x_2)\otimes\mathbf{1}\,\varepsilon_C(\phi(x_3))+\cdots
\end{aligned}$$

which are equal because of the properties of the counit ε_C. For a more rigorous calculus, one can show by induction that for $n\in\mathbb{N}$,

$$\Delta\circ\mu^{(n)}=\sum_{i=0}^{n}(\mu^{(i)}\otimes\mu^{(n-i)})\circ(id^{\otimes i}\otimes\Delta\otimes id^{\otimes n-i})-\sum_{i=0}^{n-1}\mu^{(i)}\otimes\mu^{(n-1-i)}$$

(whit $\mu^{(0)}=id$) and then compute

$$\begin{aligned}
\Delta\circ\overline{\phi}&=\Delta\circ\sum_{n\in\mathbb{N}}\phi^{\star n}=\sum_{n\in\mathbb{N}}\Delta\circ\mu^{(n-1)}\circ\phi^{\otimes n}\circ\Delta_C^{(n-1)}\\
&=\sum_{n\in\mathbb{N}}\left(\sum_{i=0}^{n-1}(\mu^{(i)}\otimes\mu^{(n-1-i)})\circ(id^{\otimes i}\otimes\Delta\otimes id^{\otimes n-1-i})\right.\\
&\qquad\left.-\sum_{i=0}^{n-2}\mu^{(i)}\otimes\mu^{(n-2-i)}\right)\circ\phi^{\otimes n}\circ\Delta_C^{(n-1)}
\end{aligned}$$

since $\operatorname{Im}\phi\subset V$, $\Delta\circ\phi=\phi\otimes\mathbf{1}+\mathbf{1}\otimes\phi$

$$\begin{aligned}
&=\sum_{n\in\mathbb{N}}\left(\sum_{i=0}^{n-1}(\mu^{(i)}\otimes\mu^{(n-1-i)})\circ(\phi^{\otimes i}\otimes(\phi\otimes\mathbf{1}+\mathbf{1}\otimes\phi)\otimes\phi^{\otimes n-1-i}\right.\\
&\qquad\left.-\sum_{i=0}^{n-2}\mu^{(i)}\otimes\mu^{(n-2-i)}\circ\phi^{\otimes n}\right)\circ\Delta_C^{(n-1)}\\
&=\sum_{n\in\mathbb{N}}\left(\sum_{i=0}^{n-1}\mu^{(i)}\circ\phi^{\otimes i+1}\circ\Delta_C^{(i)}\otimes\mu^{(n-2-i)}\circ\phi^{\otimes n-1-i}\circ\Delta_C^{(n-2-i)}\circ\Delta_C\right.\\
&\qquad+\sum_{i=0}^{n-1}\mu^{(i-1)}\circ\phi^{\otimes i}\circ\Delta_C^{(i-1)}\otimes\mu^{(n-1-i)}\circ\phi^{\otimes n-i}\circ\Delta_C^{(n-1-i)}\circ\Delta_C\\
&\qquad\left.-\sum_{i=0}^{n-2}\mu^{(i)}\circ\phi^{\otimes i+1}\circ\Delta_C^{(i)}\otimes\mu^{(n-2-i)}\circ\phi^{\otimes n-1-i}\circ\Delta_C^{(n-2-i)}\circ\Delta_C\right)\\
&=\sum_{n\in\mathbb{N}}\left(\sum_{i=0}^{n}\mu^{(i-1)}\circ\phi^{\otimes i}\circ\Delta_C^{(i-1)}\otimes\mu^{(n-1-i)}\circ\phi^{\otimes n-1}\circ\Delta_C^{(n-1-i)}\right)\circ\Delta_C
\end{aligned}$$

$$= \left(\sum_{n\in\mathbb{N}}\sum_{i=0}^{n} \phi^{\star i}\otimes\phi^{\star n-i}\right)\circ\Delta_C = \left(\sum_{r\in\mathbb{N}}\phi^{\star r}\otimes\sum_{s\in\mathbb{N}}\phi^{\star s}\right)\circ\Delta_C$$
$$= (\overline{\phi}\otimes\overline{\phi})\circ\Delta_C,$$

so $\overline{\phi} = \sum_{n\in\mathbb{N}}\phi^{\star n}$ fits. To prove the uniqueness, denote $\overline{\phi} = \sum_{n\in\mathbb{N}}\overline{\phi}_n$, with $\operatorname{Im}\overline{\phi}_n \subset V^{\otimes n}$. We have

$$(\overline{\phi}\otimes\overline{\phi})\circ\Delta_C = \Delta\circ\overline{\phi} \Rightarrow \mu\circ(\overline{\phi}\otimes\overline{\phi})\circ\Delta_C = \mu\circ\Delta\circ\overline{\phi}.$$

Since $(\mu\circ\Delta)(x_1\cdots x_k) = (k+1)x_1\cdots x_k$, this last equation reads $\overline{\phi}\star\overline{\phi} = \sum_{n\in\mathbb{N}}(n+1)\overline{\phi}_n$. So for all $n\in\mathbb{N}$, since $\overline{\phi}_0 = \mathbf{1}\,\varepsilon_C$,

$$\sum_{i=0}^{n}\overline{\phi}_i\star\overline{\phi}_{n-i} = (n+1)\overline{\phi}_n \Leftrightarrow (n-1)\overline{\phi}_n = \sum_{i=1}^{n-1}\overline{\phi}_i\star\overline{\phi}_{n-i}.$$

By induction, supposing $\overline{\phi}_k = \phi^{\star k}$ for $0\leqslant k\leqslant n$, $n\overline{\phi}_{n+1} = \sum_{i=1}^{n}\overline{\phi}_i\star\overline{\phi}_{n+1-i} = \sum_{i=1}^{n}\phi^{\star n+1} = n\phi^{\star n+1}$, so we also have $\overline{\phi}_{n+1} = \phi^{\star n+1}$ and thus $\overline{\phi}_n = \phi^{\star n}$ for all $n\in\mathbb{N}$. So $\overline{\phi} = \sum_{n\in\mathbb{N}}\phi^{\star n}$ is uniquely determined by ϕ. □

The coalgebra morphism $\overline{\phi}$ is said to be coinduced by $\phi : C^+ \to V$ and is computed as $\overline{\phi} = \sum_{n\in\mathbb{N}}\phi^{\star n}$, the geometric serie using the convolution with respect to the free multiplication μ and the comultiplication Δ_C.
Two coalgebra morphisms $\Phi,\Psi : \mathcal{T}V \leftarrow C$ are equal if and only if there components $\mathrm{pr}_V\circ\Phi$ and $\mathrm{pr}_V\circ\Psi$ are equal. For example, μ_{sh} is coinduced by $\mathrm{pr}\otimes\varepsilon + \varepsilon\otimes\mathrm{pr}$, see equation (1.1.10).

Proposition 1.1.5 *Let $d : C^+ \to V$ be a linear map of degree $j\in\mathbb{Z}$. There exists a unique graded coderivation of degree j along $\overline{\phi}$, noted $\overline{d} : C \to \mathcal{T}V$, i.e. $\Delta\circ\overline{d} = (\overline{d}\otimes\overline{\phi} + \overline{\phi}\otimes\overline{d})\circ\Delta_C$, such that $pr_V\circ\overline{d} = d$.*

Proof. Using similar arguments, $\overline{d} = \overline{\phi}\star d\star\overline{\phi}$ fits. Indeed, for $x\in C^+$, in Sweedler notation, the following expressions are equal.

$$\Delta\circ\overline{d}(x) = \Delta\Big(d(x) + d(x_1)\phi(x_2) + \phi(x_1)d(x_2)$$
$$+ d(x_1)\phi(x_2)\phi(x_3) + \phi(x_1)d(x_2)\phi(x_3) + \phi(x_1)\phi(x_2)d(x_3) + \cdots\Big)$$
$$= d(x)\otimes\mathbf{1} + \mathbf{1}\otimes d(x) + \Delta(d(x_1)\phi(x_2)) + \Delta(\phi(x_1)d(x_2)) + \cdots$$

$$(\overline{d}\otimes\overline{\phi} + \overline{\phi}\otimes\overline{d})\circ\Delta_C(x) = d(x_1)\otimes\mathbf{1}\,\varepsilon_C(x_2) + \mathbf{1}\,\varepsilon_C(x_1)\otimes d(x_2)$$
$$+ \mathbf{1}\otimes d(x_1)\phi(x_2) + d(x_1)\otimes\phi(x_2) + d(x_1)\phi(x_2)\otimes\mathbf{1}$$
$$+ \mathbf{1}\otimes\phi(x_1)d(x_2) + \phi(x_1)\otimes d(x_2) + \phi(x_1)d(x_2)\otimes\mathbf{1} + \cdots$$

As before, the composition with μ

$$\Delta \circ \overline{d} = (\overline{d} \otimes \overline{\phi} + \overline{\phi} \otimes \overline{d}) \circ \Delta_C \Rightarrow \mu \circ \Delta \circ \overline{d} = \overline{d} \star \overline{\phi} + \overline{\phi} \star \overline{d}$$

gives the uniqueness of $\overline{d}$. □

Conversely, any graded coderivation $D : C \to \mathcal{T}V$ is determined by its component $d := \mathrm{pr}_V \circ D$. Moreover, d is determined by its restrictions $d_k := d|_{V^{\otimes k}}$ for any nonnegative integer k.

Taking the coalgebra $C = (\mathcal{T}V, \Delta)$ and considering coderivations along $\overline{\phi} = id_{\mathcal{T}V}$, we note for $d_1, d_2 : \mathcal{T}V \to V$

$$d_1 \circ_G d_2 := d_1 \circ \overline{d_2} = d_1 \circ (id_{\mathcal{T}V} \star d_2 \star id_{\mathcal{T}V}) = d_1 \circ \sum_{i,j=0}^{\infty} id^{\otimes i} \otimes d_2 \otimes id^{\otimes j} \; : \; \mathcal{T}V \to V \tag{1.1.7}$$

the *graded Gerstenhaber multiplication*. So, the graded commutator $[\overline{d_1}, \overline{d_2}] = \overline{d_1} \circ \overline{d_2} - (-1)^{|d_1||d_2|} \overline{d_2} \circ \overline{d_1}$ is a coderivation along $id_{\mathcal{T}V}$ which is coinduced by $\mathrm{pr}_V [\overline{d_1}, \overline{d_2}] = d_1 \circ_G d_2 - (-1)^{|d_1||d_2|} d_2 \circ d_1 =: [d_1, d_2]_G$, the *graded Gerstenhaber bracket* of d_1 and d_2. There is then equality between

$$d \circ_G \mathrm{pr}_V [\overline{d_1}, \overline{d_2}] = d \circ [\overline{d_1}, \overline{d_2}] = (d \circ_G d_1) \circ_G d_2 - (-1)^{|d_1||d_2|} (d \circ_G d_2) \circ_G d_1$$

and

$$d \circ_G [d_1, d_2]_G = d \circ_G (d_1 \circ_G d_2) - (-1)^{|d_1||d_2|} d \circ_G (d_2 \circ_G d_1),$$

known as the Gerstenhaber identity

$$(d \circ_G d_1) \circ_G d_2 - (-1)^{|d_1||d_2|} (d \circ_G d_2) \circ_G d_1 = d \circ_G [d_1, d_2]_G. \tag{1.1.8}$$

So there is a graded pre-Lie identity for $\circ_G$, so $(\mathrm{Hom}(\mathcal{T}V, V), [\ ,\ ]_G)$ is a graded Lie algebra.

The structures $\circ_G$ and $[\ ,\ ]_G$ were first defined by Gerstenhaber in [Ger63] for $V[1]$ where the graded space was $V = V^0$.

Lemma 1.1.6 *Let $\mu : V \otimes V \to V$ be a graded associative multiplication (of degree 0). Then the shifted map $d = \mu[1] : V[1] \otimes V[1] \to V[1]$ is of degree 1, and the associativity of μ is equivalent to $d \circ_G d = 0$.*

Proof. We have that

$$d : V[1]^{\otimes 2} \to V[1]$$
$$d(a \otimes b) = s^{-1} \circ \mu \circ (s \otimes s)(a \otimes b) = (-1)^{|a|} s^{-1} \mu(s(a) \otimes s(b)) = (-1)^{|a|} ab,$$

and for $a,b,c \in V[1]$,

$$
\begin{aligned}
(d \circ_G d)(a \otimes b \otimes c) &= (\mu[1] \circ \overline{\mu[1]})(a \otimes b \otimes c) \\
&= \mu[1](\mu[1] \otimes id + id \otimes \mu[1])(a \otimes b \otimes c) \\
&= \mu[1](\mu[1](a \otimes b) \otimes c) + (-1)^{|a|}\mu[1](a \otimes \mu[1](b \otimes c)) \\
&= (-1)^{|a|}\mu[1](ab \otimes c) + (-1)^{|a|+|b|}\mu[1](a \otimes bc) \\
&= (-1)^{|a|+|ab|}(ab)c + (-1)^{2|a|+|b|}a(bc)
\end{aligned}
$$

since $|ab| = |s^{-1}(sa\, sb)| = |a| + |b| + 1$

$$
\begin{aligned}
&= (-1)^{2|a|+|b|}\big(a(bc) - (ab)c\big) \\
&= (-1)^{|b|-1}\mathfrak{as}_{\mu[1]}(a,b,c) \\
&= 0.
\end{aligned}
$$

□

1.1.4 Symmetric bialgebra

The *graded symmetric bialgebra* on V, $\mathcal{S}\,V = \oplus_{n\in\mathbb{N}}\mathcal{S}^n\,V$ is defined as the quotient of the free algebra $\mathcal{T}\,V$ by the two-sided ideal generated by all elements $xy - (-1)^{|x||y|}yx$ in $\mathcal{T}\,V$ with $x \in V^{|x|}$, $y \in V^{|y|}$. The resulting associative multiplication, the *shuffle multiplication* $\bullet$ is graded commutative, *i.e.* for two homogenous elements $a,b \in \mathcal{S}\,V$ we have $a \bullet b = (-1)^{|a||b|}b \bullet a$, and has for unit element $\mathbf{1}$. So the quotient $\mathcal{S}\,V$ is a graded commutative associative algebra.

Moreover, the first comultiplication Δ_{sh} factors through the quotient and define on $\mathcal{S}\,V$ a graded cocommutative comultiplication, also noted Δ_{sh}. The space $\mathcal{S}\,V$ is then a graded commutative cocommutative bialgebra. It is the free graded commutative algebra generated by V.

For an integer n, a permutation σ from the symmetric group S_n and $\xi = x_1 \cdots x_n \in V^{\otimes n}$, denote $\xi^\sigma = x_{\sigma(1)} \cdots x_{\sigma(n)}$ the usual right action from S_n on $\mathcal{T}\,V$. Defining the *graded signature from σ with respect to ξ* as

$$
\begin{aligned}
e(x_1 \cdots x_n, \sigma) &:= \prod_{1\leqslant i<j\leqslant n} \frac{\sigma(i) + (-1)^{|x_{\sigma(i)}||x_{\sigma(j)}|}\sigma(j)}{i + (-1)^{|x_i||x_j|}j} \\
&= \prod_{i<j \text{ and } \sigma(i)>\sigma(j)} (-1)^{|x_{\sigma(i)}||x_{\sigma(j)}|},
\end{aligned}
$$

there is a graded right action $\xi^{\cdot\sigma} = e(\xi,\sigma)\xi^\sigma$ from S_n on $V^{\otimes n}$, because $e(\xi,\sigma\tau) = e(\xi,\sigma)e(\xi^\sigma,\tau)$.

Using this action, we can give an explicit formula of the shuffle multiplication. Note $Sh(k,n-k)$ the set of shuffle permutations, *i.e.* permutations $\sigma \in S_n$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$. Then

$$(x_1\cdots x_k)\bullet(x_{k+1}\cdots x_n)=\sum_{\sigma^{-1}\in Sh(k,n-k)} e(x_1\cdots x_n,\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)}. \tag{1.1.9}$$

For example, we have $Sh(1,1)=S_2=\{id,(1\ 2)\}$, and $e(x_1x_2,id)=1$, $e(x_1x_2,(1\ 2))=(-1)^{|x_2||x_1|}$, so

$$x_1\bullet x_2=x_1x_2+(-1)^{|x_2||x_1|}x_2x_1.$$

$$\begin{gathered}
Sh(2,1)=\left\{id,\left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)\right\}=\{id,(2\ 3)^{-1},(1\ 3\ 2)^{-1}\}\\
e(x_1x_2x_3,(23))=(-1)^{|x_3||x_2|},\qquad e(x_1x_2x_3,(132))=(-1)^{|x_3|(|x_1|+|x_2|)}\\
x_1x_2\bullet x_3=x_1x_2x_3+(-1)^{|x_3||x_2|}x_1x_3x_2+(-1)^{|x_3|(|x_1|+|x_2|)}x_3x_1x_2
\end{gathered}$$

$$\begin{gathered}
Sh(1,2)=\left\{id,\left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)\right\}=\{id,(1\ 2)^{-1},(1\ 2\ 3)^{-1}\}\\
e(x_1x_2x_3,(12))=(-1)^{|x_2||x_1|},\qquad e(x_1x_2x_3,(123))=(-1)^{|x_1|(|x_2|+|x_3|)}\\
x_1\bullet x_2x_3=x_1x_2x_3+(-1)^{|x_2||x_1|}x_2x_1x_3+(-1)^{|x_1|(|x_2|+|x_3|)}x_2x_3x_1
\end{gathered}$$

The shuffle multiplication can be computed recursively as follow. For $\lambda\in\mathbb{K}$ and $x_1\cdots x_n\in\mathcal{T}V$, we have $\lambda\bullet x_1\cdots x_n=\lambda x_1\cdots x_n=x_1\cdots x_n\bullet\lambda$ and

$$\begin{aligned}
x_1\cdots x_k\bullet x_{k+1}\cdots x_n&=x_1\,(x_2\cdots x_k\bullet x_{k+1}\cdots x_n)\\
&\quad+(-1)^{|x_{k+1}|(|x_1|+\cdots+|x_k|)}x_{k+1}\,(x_1\cdots x_k\bullet x_{k+2}\cdots x_n).
\end{aligned}$$

The shuffle multiplication is coinduced by the map $\mathsf{pr}_V\otimes\varepsilon+\varepsilon\otimes\mathsf{pr}_V$.

$$\begin{gathered}
\mu_{sh}=\overline{\mathsf{pr}_V\otimes\varepsilon+\varepsilon\otimes\mathsf{pr}_V}=\sum_{n\in\mathbb{N}}(\mathsf{pr}_V\otimes\varepsilon+\varepsilon\otimes\mathsf{pr}_V)^{\star n}\\
\Leftrightarrow \mathsf{pr}_V\circ\mu_{sh}=\mathsf{pr}_V\otimes\varepsilon+\varepsilon\otimes\mathsf{pr}_V
\end{gathered}\tag{1.1.10}$$

Indeed, noting $\xi=x_1\cdots x_k$ et $\eta=x_{k+1}\cdots x_n$, we obtain by induction

$$\begin{aligned}
&\sum_{n\in\mathbb{N}}\mu^{(n-1)}\circ(\mathsf{pr}_V\otimes\varepsilon+\varepsilon\otimes\mathsf{pr}_V)^{\otimes n}\circ\Delta^{[2](n-1)}(x_1\cdots x_k\otimes x_{k+1}\cdots x_n)\\
&\quad=\sum_{n\in\mathbb{N}}\mu^{(n-1)}\circ(\mathsf{pr}_V\otimes\varepsilon+\varepsilon\otimes\mathsf{pr}_V)^{\otimes n}(\xi_1\otimes\eta_1\otimes\xi_{\text{remain}}\otimes\eta_{\text{remain}})\\
&\quad=\mathsf{pr}_V(\xi_1)\varepsilon(\eta_1)(\xi_{\text{remain}}\bullet\eta_{\text{remain}})+(-1)^{|\eta_1||\xi|}\varepsilon(\xi_1)\,\mathsf{pr}_V(\eta_1)(\xi_{\text{remain}}\bullet\eta_{\text{remain}})\\
&\quad=\mathsf{pr}_V(\xi_1)(\xi_{\text{remain}}\bullet\eta)+(-1)^{|\eta_1||\xi|}\,\mathsf{pr}_V(\eta_1)(\xi\bullet\eta_{\text{remain}})\\
&\quad=x_1\,(x_2\cdots x_k\bullet x_{k+1}\cdots x_n)\\
&\qquad+(-1)^{|x_{k+1}|(|x_1|+\cdots+|x_k|)}x_{k+1}\,(x_1\cdots x_k\bullet x_{k+2}\cdots x_n).
\end{aligned}$$

The associativity of μ_{sh} follows easily: to prove that $\mu_{sh} \circ (\mu_{sh} \otimes id) = \mu_{sh} \circ (id \otimes \mu_{sh})$, it suffices to have equality on projections on V.

$$\begin{aligned}
\mathrm{pr}_V \circ \mu_{sh} \circ (\mu_{sh} \otimes id) &= (\mathrm{pr}_V \otimes \varepsilon + \varepsilon \otimes \mathrm{pr}_V) \circ (\mu_{sh} \otimes id) \\
&= (\mathrm{pr}_V \otimes \varepsilon + \varepsilon \otimes \mathrm{pr}_V) \otimes \varepsilon + \varepsilon \otimes \varepsilon \otimes \mathrm{pr}_V \\
&= \mathrm{pr}_V \otimes \varepsilon \otimes \varepsilon + \varepsilon \otimes \mathrm{pr}_V \otimes \varepsilon + \varepsilon \otimes \varepsilon \otimes \mathrm{pr}_V \\
&= \mathrm{pr}_V \otimes \varepsilon \otimes \varepsilon + \varepsilon \otimes (\mathrm{pr}_V \otimes \varepsilon + \varepsilon \otimes \mathrm{pr}_V) \\
\mathrm{pr}_V \circ \mu_{sh} \circ (id \otimes \mu_{sh}) &= (\mathrm{pr}_V \otimes \varepsilon + \varepsilon \otimes \mathrm{pr}_V) \circ (id \otimes \mu_{sh})
\end{aligned}$$

Like for the tensor coalgebra, the graded cocommutative coalgebra $\mathcal{S}\,V$ is cofree in the category $\mathcal{CS}_{AN}$ of graded cocommutatives augmented coalgebras with C^+ nilpotent, and the coinduction of morphisms and coderivations is done as in the diagram (1.1.6), with $\mathcal{T}\,V$ replaced by $\mathcal{S}\,V$. The shuffle multiplication μ_{sh} of $\mathcal{S}\,V$ can be used instead of the multiplicatoin μ from $\mathcal{T}\,V$ to obtain another convolution $\tilde{\star}$. Coinduced morphisms and coderivations are computed as $\overline{\phi} = e^{\tilde{\star}\phi}$ instead of the geometric serie, and $\overline{d} = d\tilde{\star}e^{\tilde{\star}\phi}$ instead of $\overline{\phi} \star d \star \overline{\phi}$.

Taking the coalgebra $C = (\mathcal{S}\,V, \Delta_{sh})$ and considering coderivations along $\overline{\phi} = id_{\mathcal{S}\,V}$, we note for $d_1, d_2 : \mathcal{S}\,V \to V$

$$d_1 \circ_{NR} d_2 := d_1 \circ \overline{d_2} = d_1 \circ (d_2 \tilde{\star} id_{\mathcal{S}\,V}) \; : \; \mathcal{S}\,V \to V \tag{1.1.11}$$

the *graded Nijenhuis-Richardson multiplication*. For $d_1 \in \mathrm{Hom}(\mathcal{S}^r\,V, V)$ and $d_2 \in \mathrm{Hom}(\mathcal{S}^t\,V, V)$, it computes as

$$\begin{aligned}
&(d_1 \circ_{NR} d_2)(x_1 \bullet \cdots \bullet x_{r+t-1}) = \\
&\sum_{1 \leqslant i_1 < \cdots < i_t \leqslant r+t-1} \prod_{p=1}^{t} (-1)^{|x_{i_p}|\left(|x_1| + \cdots + \widehat{|x_{i_1}|} + \cdots + \widehat{|x_{i_{p-1}}|} + \cdots + |x_{i_p - 1}|\right)} \\
&d_1(d_2(x_{i_1} \bullet \cdots \bullet x_{i_t}) \bullet x_1 \bullet \cdots \bullet \widehat{x_{i_1}} \bullet \cdots \bullet \widehat{x_{i_t}} \bullet \cdots \bullet x_{r+t-1})
\end{aligned} \tag{1.1.12}$$

for $x_1, \ldots, x_{r+t-1} \in V$, where $\widehat{\ }$ indicates that the argument is omitted.

As before, this composition satisfies the identity (1.1.8) with $\circ_G$ replaced by $\circ_{NR}$, thus giving $(\mathrm{Hom}(\mathcal{S}\,V, V), [\ ,\ ]_{NR})$ a structure of a graded Lie algebra, where $[d_1, d_2]_{NR} := d_1 \circ_{NR} d_2 - (-1)^{|d_1||d_2|} d_2 \circ_{NR} d_1$ is the graded Nijenhuis-Richardson bracket.

The structures $\circ_{NR}$ et $[\ ,\ ]_{NR}$ were defined in [NR66] for $V[1]$ where the graded space was $V = V^0$.

Lemma 1.1.7 *Let $[\ ,\ ] : V \otimes V \to V$ be a graded Lie bracket (of degree 0). Then the shifted map $d = [\ ,\ ][1] : V[1] \bullet V[1] \to V[1]$ is symmetric of degree 1, and the Jacobi identity for $[\ ,\ ]$ is equivalent to $d \circ_{NR} d = 0$.*

Proof. We have that

$$d : V[1]^{\bullet 2} \to V[1]$$
$$d(a \bullet b) = s^{-1} \circ [\ ,\ ] \circ (s \otimes s)(a \otimes b) = (-1)^{|a|} s^{-1}[s(a), s(b)] = (-1)^{|a|}[a,b].$$

The map $d = [\ ,\ ][1]$ is symmetric of degree 1, because

$$\begin{aligned} d(b \bullet a) &= (-1)^{|b|}[b,a] = -(-1)^{|b|+(|a|+1)(|b|+1)}[a,b] \\ &= (-1)^{|a||b|}(-1)^{|a|}[a,b] = (-1)^{|a||b|} d(a \bullet b), \end{aligned}$$

and for $x, y, z \in V[1]$,

$$\begin{aligned} &(d \circ_{NR} d)(x \bullet y \bullet z) \\ &= d(d(x \bullet y) \bullet z) + (-1)^{|y||z|} d(d(x \bullet z) \bullet y) + (-1)^{|x|(|y|+|z|)} d(d(y \bullet z) \bullet x) \\ &= (-1)^{|x|} d([x,y] \bullet z) + (-1)^{|y||z|+|x|} d([x,z] \bullet y) \\ &\qquad + (-1)^{|x|(|y|+|z|)+|y|} d([y,z] \bullet x) \\ &= (-1)^{2|x|+|y|+1}[[x,y],z] + (-1)^{|y||z|+2|x|+|z|+1}[[x,z],y] \\ &\qquad + (-1)^{|x|(|y|+|z|)+2|y|+|z|+1}[[y,z],x] \\ &= (-1)^{|y|+1}[[x,y],z] + (-1)^{|y||z|+|z|+2+(|x|+1)(|z|+1)}[[z,x],y] \\ &\qquad + (-1)^{|x|(|y|+|z|)+|z|+1}[[y,z],x] \\ &= (-1)^{|y|+1}(-1)^{(|x|+1)(|z|+1)} \\ &\qquad \Big((-1)^{(|x|+1)(|z|+1)}[[x,y],z] + (-1)^{(|y|+1)(|x|+1)}[[y,z],x] \\ &\qquad\qquad + (-1)^{(|z|+1)(|y|+1)}[[z,x],y] \Big) \\ &= 0 \end{aligned}$$

since $[\ ,\ ]$ satisfies the graded Jacobi identity in V. (The degree of $x \in V[1]$ is $|x|+1$ where $|x|$ is the degree of $x \in V$). □

1.1.5 Universal enveloping algebras

Let $\mathbb{K}$ be a commutative ring and $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Recall that a (left) $\mathfrak{g}$-representation of $\mathfrak{g}$ is a $\mathbb{K}$-module $\mathcal{M}$ and a $\mathbb{K}$-homomorphism $\mathfrak{g} \otimes \mathcal{M} \to \mathcal{M}$, $x \otimes a \mapsto xa$ such that $x(ya) - y(xa) = [x,y]a$. To each Lie algebra $\mathfrak{g}$, an associative $\mathbb{K}$-algebra $\mathcal{U}(\mathfrak{g})$ is associated such that every (left) $\mathfrak{g}$-representation may be viewed as (left) $\mathcal{U}(\mathfrak{g})$-representation and vice-versa. The algebra $\mathcal{U}(\mathfrak{g})$ is constructed as follows.

Let $\mathcal{T}\mathfrak{g}$ be the tensor algebra of the $\mathbb{K}$-module $\mathfrak{g}$,

$$\mathcal{T}\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \mathcal{T}^n\,\mathfrak{g} \quad \text{where} \quad \mathcal{T}^n\,\mathfrak{g} = \mathfrak{g}\otimes\mathfrak{g}\otimes\cdots\otimes\mathfrak{g} \quad \text{(n times)}.$$

In particular $\mathcal{T}^0\,\mathfrak{g} = \mathbb{K}\mathbf{1}$ and $\mathcal{T}^1\,\mathfrak{g} = \mathfrak{g}$. The multiplication in $\mathcal{T}\mathfrak{g}$ is the tensor product. Every $\mathbb{K}$-linear map $\mathfrak{g}\otimes\mathcal{M}\to\mathcal{M}$ has a unique extension to a a $\mathcal{T}\mathfrak{g}$-module map $\mathcal{T}\mathfrak{g}\otimes\mathcal{M}\to\mathcal{M}$. If $\mathfrak{g}\otimes\mathcal{M}\to\mathcal{M}$ is a $\mathfrak{g}$-module then the vector space $\mathfrak{g}$ inside $\mathcal{T}\mathfrak{g}$ is in general not a Lie subalgebra being represented on $\mathcal{M}$. This is remedied if and only if the elements of $\mathcal{T}\mathfrak{g}$ of the form $x\otimes y - y\otimes x - [x,y]$ where $x,y\in\mathfrak{g}$, are sent to 0. Consequently, one is led to introduce the two-sided ideal I generated by the elements $x\otimes y - y\otimes x - [x,y]$ where $x,y\in\mathfrak{g}$. The enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is thus defined as $\mathcal{T}\mathfrak{g}/I$. It follows that $\mathfrak{g}$-representations and $\mathcal{U}(\mathfrak{g})$-modules may be identified. Recall that every $\mathcal{U}(\mathfrak{g})$-bimodule $\mathcal{M}$ is a $\mathfrak{g}$-module by $(x,m)\to xm - mx$, denoted by $\mathcal{M}_a$.

Assume that $\mathfrak{g}$ is a free Lie algebra. Let $\{x_i\}$ be a fixed basis of $\mathfrak{g}$ and y_i be the image of x_i by the map $\mathfrak{g}\to\mathcal{T}\mathfrak{g}\to\mathcal{U}(\mathfrak{g})$. We set $y_I = y_{i_1}\cdots y_{i_p}$ with I a finite sequence of indices $i_1,\ldots,i_p$ and $y_I = 1$ if $I=\emptyset$. The Poincaré-Birkhoff-Witt Theorem insures that the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is generated by the elements y_I corresponding to increasing sequences I.

We denote by $\mathcal{S}\,V$ the symmetric algebra over a $\mathbb{K}$-module V. If $\mathbb{Q}\subset\mathbb{K}$, then there exists a canonical bijection between $\mathcal{S}\,\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ which is a $\mathfrak{g}$-module isomorphism between $\mathcal{S}\,\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g}_a)$ (see [Dix74, pp.78-79] or [LV12, Section 3.6.13]).

1.2 Cohomology and deformations

We briefly review the notions of Hochschild and Chevalley-Eilenberg cohomology. The formal deformation of rings and algebras was introduced by M. Gerstenhaber in 1964 ([Ger66]). He gave a tool to deform algebraic structures based on formal power series. The interest on deformations has grown with the development of quantum groups related to quantum mechanics ([BFF+78]). Examples of quantum groups may be obtained as Hopf algebra deformations of the enveloping algebra of a Lie algebra.

1.2.1 Hochschild cohomology

Let $(\mathcal{A},\mu)$ be an associative algebra over the field $\mathbb{K}$. For each positive integer n we define the space of Hochschild cochains of degree n with coefficients in $\mathcal{M}$, $\mathrm{C_H}^n(\mathcal{A},\mathcal{M}) := \mathrm{Hom}(\mathcal{A}^{\otimes n},\mathcal{M})$, and $\mathrm{C_H}^0(\mathcal{A},\mathcal{M}) := \mathcal{M}$. We define $\mathrm{C_H}(\mathcal{A},\mathcal{M}) := \oplus_{n=0}^{\infty}\mathrm{C_H}^n(\mathcal{A},\mathcal{M})$ which is

a $\mathbb{Z}$-graded vector space by setting ${\mathrm{C_H}}^n(\mathcal{A},\mathcal{M}) := \{0\}$ for all strictly negative integers n.
For $f \in {\mathrm{C_H}}^n(\mathcal{A},\mathcal{M})$, we define the n^{th} Hochschild coboundary operator by

$$\begin{aligned}(\delta_H^n f)(a_1,\ldots,a_{n+1}) = {} & a_1 f(a_2,\ldots,a_{n+1}) \\ & + \sum_{i=1}^{n}(-1)^i f(a_1,\ldots,a_{i-1},a_i a_{i+1},a_{i+2},\ldots,a_{n+1}) \\ & + (-1)^{n+1} f(a_1,\ldots,a_n)a_{n+1}.\end{aligned} \tag{1.2.1}$$

For each nonnegative integer n denote by ${\mathrm{H_H}}^n(\mathcal{A},\mathcal{M})$ the n^{th} Hochschild cohomology group with coefficients in $\mathcal{M}$,

$${\mathrm{H_H}}^n(\mathcal{A},\mathcal{M}) := \frac{\mathrm{Ker}(\delta_H^n : {\mathrm{C_H}}^n(\mathcal{A},\mathcal{M}) \to {\mathrm{C_H}}^{n+1}(\mathcal{A},\mathcal{M}))}{\mathrm{Im}(\delta_{n-1} : {\mathrm{C_H}}^{n-1}(\mathcal{A},\mathcal{M}) \to {\mathrm{C_H}}^n(\mathcal{A},\mathcal{M}))} := \frac{Z\,{\mathrm{C_H}}^n(\mathcal{A},\mathcal{M})}{B\,{\mathrm{C_H}}^n(\mathcal{A},\mathcal{M})} \tag{1.2.2}$$

where again we set ${\mathrm{H_H}}^n(\mathcal{A},\mathcal{M}) := \{0\}$ for all strictly negative integers n. We also define

$$\mathrm{H_H}(\mathcal{A},\mathcal{M}) := \bigoplus_{n\in\mathbb{Z}} {\mathrm{H_H}}^n(\mathcal{A},\mathcal{M}) := Z\,\mathrm{C_H}(\mathcal{A},\mathcal{M})/B\,\mathrm{C_H}(\mathcal{A},\mathcal{M}).$$

1.2.2 Chevalley-Eilenberg cohomology

Let $\mathfrak{g}$ a Lie algebra equipped with a representation $\rho : \mathfrak{g} \to \mathrm{Hom}(V,V)$. The Chevalley-Eilenberg space of cochains of degree n is given by ${\mathrm{C_{CE}}}^n(\mathfrak{g},V) := \mathrm{Hom}(\bigwedge^k \mathfrak{g},V)$ for each $n \in \mathbb{N}^\star$, ${\mathrm{C_{CE}}}^0(\mathfrak{g},V) := \mathfrak{g}$ and ${\mathrm{C_{CE}}}^n(\mathfrak{g},V) := \{0\}$ for all strictly negative integers n.
For $f \in {\mathrm{C_{CE}}}^n(\mathfrak{g},V)$, we define the n^{th} Chevalley-Eilenberg coboundary operator by

$$\begin{aligned}(\delta_{CE}^n f)(x_0,\ldots,x_n) = {} & \sum_{i=0}^{n}(-1)^i \rho(x_i) f(x_0,\ldots,\hat{x}_i,\ldots,x_n) \\ & + \sum_{0\leqslant i<j\leqslant n} (-1)^{i+j} f([x_i,x_j],x_0,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_n)\end{aligned} \tag{1.2.3}$$

For each nonnegative integer n denote by ${\mathrm{H_{CE}}}^n(\mathfrak{g},V)$ the n^{th} Chevalley-Eilenberg cohomology group with coefficients in V,

$${\mathrm{H_{CE}}}^n(\mathfrak{g},V) := \frac{\mathrm{Ker}(\delta_{CE}^n : {\mathrm{C_{CE}}}^n(\mathfrak{g},V) \to {\mathrm{C_{CE}}}^{n+1}(\mathfrak{g},V))}{\mathrm{Im}(\delta_{CE}^{n-1} : {\mathrm{C_{CE}}}^{n-1}(\mathfrak{g},V) \to {\mathrm{C_{CE}}}^n(\mathfrak{g},V))} := \frac{Z\,{\mathrm{C_{CE}}}^n(\mathfrak{g},V)}{B{\mathrm{C_{CE}}}^n(\mathfrak{g},V)} \tag{1.2.4}$$

where again we set ${\mathrm{H_{CE}}}^n(\mathfrak{g},V) := \{0\}$ for all strictly negative integers n. We also define

$$\mathrm{H_{CE}}(\mathfrak{g},V) := \bigoplus_{n\in\mathbb{Z}} {\mathrm{H_{CE}}}^n(\mathfrak{g},V)^n := ZC_{CE}^n(\mathfrak{g},V)/BC_{CE}^n(\mathfrak{g},V).$$

1.2.3 Properties

We shall need to compute Hochschild and Chevalley-Eilenberg cohomology of $\mathcal{U}(\mathfrak{g})$ and $\mathfrak{g}$, respectively, wherefore we shall cite the following two Theorems.

The classical Theorem due to H.Cartan et S.Eilenberg, ([CE56, pp.277]) gives a link between the Hochschild cohomology of an enveloping algebra with values in an $\mathcal{U}(\mathfrak{g})$-bimodule $\mathcal{M}$ (in particular $\mathcal{M} = \mathcal{U}(\mathfrak{g})$) and the Chevalley-Eilenberg cohomology of the Lie algebra with values in the same module.

Theorem 1.2.1 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$. Then*

$$\mathrm{H_H}^n(\mathcal{U}(\mathfrak{g}), \mathcal{M}) \simeq \mathrm{H_{CE}}^n(\mathfrak{g}, \mathcal{M}_a) \quad \forall n \in \mathbb{N}$$

In particular, if $\mathbb{Q} \subset \mathbb{K}$

$$\mathrm{H_H}^n(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g})) \simeq \mathrm{H_{CE}}^n(\mathfrak{g}, \mathcal{U}(\mathfrak{g}_a)) \simeq \mathrm{H_{CE}}^n(\mathfrak{g}, \mathcal{S}\,\mathfrak{g}) \quad \forall n \in \mathbb{N}$$

The *Hochschild-Serre Theorem* [HS53] gives the following factorization of the Chevalley-Eilenberg cohomology groups in the case of a decomposable solvable Lie algebra.

Theorem 1.2.2 *Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t}$ be a finite dimensional solvable Lie algebra over $\mathbb{K}$, where $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{g}$ and $\mathfrak{t}$ the supplementary subalgebra of $\mathfrak{n}$, reductive in $\mathfrak{g}$, such that the $\mathfrak{t}$-module induced on $\mathcal{U}(\mathfrak{g})_a$ is semisimple, then for all nonnegative integers p, we have*

$$\mathrm{H_{CE}}^p(\mathfrak{g}, \mathcal{U}(\mathfrak{g}_a)) \simeq \sum_{i+j=p} \mathrm{H_{CE}}^i(\mathfrak{t}, \mathbb{K}) \otimes \mathrm{H_{CE}}^j(\mathfrak{n}, \mathcal{U}(\mathfrak{g}_a))^{\mathfrak{t}}.$$

where $\mathrm{H_{CE}}^j(\mathfrak{n}, \mathcal{U}(\mathfrak{g}_a))^{\mathfrak{t}}$ denotes the subspace of $\mathfrak{t}$-invariant elements.

1.2.4 Link with deformations

Let $\mathbb{K}[[t]]$ be the power series ring with coefficients in $\mathbb{K}$. For a $\mathbb{K}$-vector space E we denote by $E[[t]]$ the $\mathbb{K}[[t]]$-module of the power series with coefficients in E. Let $(\mathcal{A}, \mu_0)$ be an associative (resp. Lie) $\mathbb{K}$-algebra, then $(\mathcal{A}[[t]], \mu_0)$ is an associative (resp. Lie) $\mathbb{K}[[t]]$-algebra.

A *formal deformation* of an associative (resp. Lie algebra) $\mathcal{A}$ is an associative (resp. Lie) $\mathbb{K}[[t]]$-algebra $(\mathcal{A}[[t]], \mu)$ such that

$$\mu = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots + t^n\mu_n + \cdots,$$

where $\mu_n \in \mathrm{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$ (resp. $\mu_n \in \mathrm{Hom}(\mathcal{A} \wedge \mathcal{A}, \mathcal{A})$). Moreover, two deformations $(\mathcal{A}[[t]], \mu)$ and $(\mathcal{A}[[t]], \mu')$ are said to be *equivalent* if there exists a formal isomorphism

$$\varphi = \varphi_0 + \varphi_1 t + \cdots + \varphi_n t^n + \cdots,$$

with $\varphi_0 = id_{\mathcal{A}}$ (Identity map on $\mathcal{A}$) and $\varphi_n \in \mathrm{Hom}(\mathcal{A},\mathcal{A})$ such that

$$\mu'(a,b) = \varphi_t^{-1}(\mu(\varphi(a),\varphi(b)) \quad \forall a,b \in \mathcal{A}.$$

A deformation of $\mathcal{A}$ is called *trivial* if it is equivalent to $(\mathcal{A}[[t]],\mu_0)$. An associative (resp. Lie) algebra $\mathcal{A}$ is said to be *rigid* if every deformation of $\mathcal{A}$ is trivial.

There is a relation between formal deformation theory and Hochschild cohomology in the case of an associative algebra and Chevalley-Eilenberg cohomology in the case of Lie algebra. We denote by ${\mathrm{H_H}}^n(\mathcal{A},\mathcal{M})$ the n^{th} Hochschild cohomology group of an associative algebra $\mathcal{A}$ with values in the bimodule $\mathcal{M}$ and by ${\mathrm{H_{CE}}}^n(\mathfrak{g},\mathcal{M})$ the n^{th} Chevalley-Eilenberg cohomology group of a Lie algebra $\mathcal{A}$ with values in a $\mathfrak{g}$-module $\mathcal{M}$. The second Hochschild cohomology group of an associative algebra (resp. Chevalley-Eilenberg cohomology group of a Lie algebra) with values in the algebra may be interpreted as the group of infinitesimal deformations. The rigidity Theorem of Gerstenhaber [Ger66] (resp. of Nijenhuis-Richardson [NR66]) insures that if the 2^{nd} Hochschild cohomology group ${\mathrm{H_H}}^2(\mathcal{A},\mathcal{A})$ (resp. Chevalley-Eilenberg ${\mathrm{H_{CE}}}^2(\mathfrak{g},\mathfrak{g})$) of an associative algebra $\mathcal{A}$ (resp. a Lie algebra $\mathfrak{g}$) vanishes then the algebra (resp. Lie algebra) is rigid. Therefore the semisimple associative (resp. Lie) algebras are rigid because their second cohomology groups are trivial ([GS86]).

The third cohomology group corresponds to the obstructions to extend a deformation of order n to a deformation of order $n+1$ ([Ger66], [Ger63] and [NR66]).

The rigidity of n-dimensional complex rigid Lie algebras was studied by R. Carles, Y. Diakité, M. Goze and J.M. Ancochea-Bermudez. Carles and Diakité established the classification for $n \leqslant 7$ ([CD84], [Car84]), and Ancochea with Goze did the classification for solvable Lie algebras for $n = 8$ and some classes ([GAB01]). The classification of associative rigid algebras are known up to $n \leqslant 6$ (see [GM96]).

2

Kontsevich formality

Contents

Beginning with an associative algebra, its Hochschild complex can be given a graded Lie algebra structure by shift. It is also the case for its cohomology. The question is to know if one can inject the cohomology in the Hochschild complex as a graded Lie subalgebra. It is not possible in general, a more loosely condition is to have a morphism up to homotopy which leads to the notion of formality. The more spectacular application of formality is to obtain an associative formal deformation of the initial algebra. However, even if there is no formality for the preceding Lie brackets, it is always possible to modify the L_∞ structure to build a L_∞-morphism.

2.1 Definitions

Let $V = \oplus_{j\in\mathbb{Z}} V^j$ a $\mathbb{Z}$-graded vector space.

Definition 2.1.1 We call *differential graded Lie algebra* (or *dg-Lie algebra*) a triple $(V,[\ ,\ ],\delta)$ where $[\ ,\ ] : V\otimes V \to V$ is a Lie bracket and $\delta : V \to V$ a degree 1 map, the *differential*, is a derivation of degree 1 of square zero, *i.e.* satisfying

$$\delta \circ [\ ,\ ] = [\ ,\ ] \circ (\delta \otimes id_V + id_V \otimes \delta) \qquad \delta\circ\delta = 0. \tag{2.1.1}$$

We consider $V[1]$, the shifted graded vector space of V.

Definition 2.1.2 A L_∞-*structure on* V is defined to be a graded coderivation D of $\mathcal{S}(V[1])$ of degree 1 and such that $D^2 = 0$ and the restriction $\mathrm{pr}_{V[1]} D|_{\mathcal{S}^0(V[1])} = 0$, so D is a differential on $\mathcal{S}(V[1])$.
The couple (V,D) is called an L_∞-*algebra*.

Let (V,D) be an L_∞-algebra and $d := \mathrm{pr}_V D$ whence $D = \overline{d}$ in the sense of (1.1.11). For each strictly positive integer k let d_k denote the restriction $d_k := d|_{\mathcal{S}^k(V[1])}$.

In particular, a graded Lie algebra $(V,[\ ,\ ])$ is considered as a L_∞-algebra, since the coderivation $D = \overline{d}$ induced by $d = [\ ,\ ][1]$ is a L_∞-structure by Lemma 1.1.7, projecting on $V[1]$.

A L_∞-*morphism* from a L_∞-algebra (V,D) to a L_∞-algebra (V',D') is a morphism of differential graded coalgebras $\Phi : \mathcal{S}(V[1]) \to \mathcal{S}(V'[1])$, *i.e.* Φ is a morphism of graded coalgebras intertwining differentials,

$$(\Phi\otimes\Phi)\Delta_{\mathcal{S}(V[1])} = \Delta_{\mathcal{S}(V'[1])}\circ\Phi \qquad \text{and} \qquad \Phi D = D'\Phi. \tag{2.1.2}$$

Moreover, we say that Φ is a L_∞-quasi-isomorphism if its first component $\Phi_1 := \Phi_{|V[1]}$ induces an isomorphism in cohomology.

In particular, a graded Lie algebra morphism $\phi : (V,[\ ,\ ]) \to (V',[\ ,\ ]')$ defines a L_∞-algebra morphism by $\Phi = \overline{\phi}$ such that the map Φ is equal to its first component $\Phi_1 := \Phi_{|V[1]}$ which is nothing else that $\phi[1]$.

Let $(V,[\ ,\ ],\delta)$ be a dg-Lie algebra. We consider the differential D on $\mathcal{S}(V[1])$ induced by $\delta + [\ ,\ ][1]$ and the homology H of V for the differential δ. If it is again a graded Lie algebra for the induced bracket $[\ ,\ ]'$, we can consider D', the differential of $\mathcal{S}(H[1])$ induced by $[\ ,\ ]'[1]$.

Definition 2.1.3 We say that the L_∞-algebra (V,D) is *formal* if there is a L_∞-quasi-isomorphisme Φ from (H,D') to (V,D) such that $\Phi_{|H[1]}$ is an injection into the cocycles.

In the following, we specialize the notion of formality for the Hochschild complex of an associative algebra, and the Chevalley-Eilenberg complex of a Lie algebra.

2.2 Case of associative algebras

2.2.1 Differential graded Lie algebras

Let $(\mathcal{A},\mu)$ be an associative algebra over the field $\mathbb{K}$. We note $\mathfrak{A} := \oplus_{n\in\mathbb{Z}}\mathfrak{A}^n$ the space of Hochschild cochains with values in $\mathcal{A}$ as defined in Section 1.2.1. We have $\mathfrak{A}^n := {\mathrm{C_H}}^n(\mathcal{A},\mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes n},\mathcal{A})$ for $n\in\mathbb{N}^\star$, $\mathfrak{A}^0 := \mathcal{A}$ and $\mathfrak{A}^n := \{0\}$ for strictly negative integers n. In the same way, we note $\mathfrak{a}^n$ the n^{th} Hochschild cohomology group and $\mathfrak{a} = \oplus_{n\in\mathbb{Z}}\mathfrak{a}^n$.

A useful way to obtain properties of the Hochschild cohomology is to consider the Gerstenhaber multiplication

$$\circ_G : {\mathrm{C_H}}^k(\mathcal{A},\mathcal{A})\times{\mathrm{C_H}}^l(\mathcal{A},\mathcal{A})\to{\mathrm{C_H}}^{k+l-1}(\mathcal{A},\mathcal{A})$$

which is defined on elements by

$$\begin{aligned}(f\circ_G g)(a_1,\dots,a_{k+l-1}) =& \\ &\sum_{i=1}^{k}(-1)^{(i-1)(l-1)}f(a_1,\dots,a_{i-1},g(a_i,\dots,a_{i+l-1}),a_{i+l},\dots,a_{k+l-1})\end{aligned} \tag{2.2.1}$$

and the Gerstenhaber bracket, given by

$$[f,g]_G := f\circ_G g-(-1)^{(k-1)(l-1)}g\circ_G f.$$

Proposition 2.2.1 *The shifted space $(\mathfrak{A}[1],[\ ,\ ]_G)$ is a graded Lie algebra.*

We will denote it by $(\mathcal{G},[\ ,\ ]_G)$.

Proof. We have

$$\mathfrak{A}[1] = \bigoplus_{n\in\mathbb{N}}\mathrm{Hom}(\mathcal{A}[1]^{\otimes n},\mathcal{A}[1]) \simeq \mathrm{Hom}\left(\bigoplus_{n\in\mathbb{N}}\mathcal{A}[1]^{\otimes n},\mathcal{A}[1]\right) = \mathrm{Hom}(\mathcal{T}V,V)$$

where V is the underlying vector space of $\mathcal{A}[1]$, and using the Gerstenhaber identity (1.1.8), we know that $(\mathrm{Hom}(\mathcal{T}V,V),[\ ,\ ]_G)$ is a graded Lie algebra. □

The multiplication μ is an element of $\mathfrak{A}^2$, seen as $m=\mu[1]\in\mathcal{G}^1$, with $|m| = 1$. According to Lemma 1.1.6, μ is associative if and only if $[m,m]_G = 0$.

Define $b : \mathcal{G} \to \mathcal{G}$ by $b := [m,\ ]_G$. For $f \in \mathfrak{A}^k$, $b(f) = [m, f]_G = \mu[1] \circ_G f - (-1)^{|f|} f \circ_G \mu[1]$. For $a_1, \ldots, a_{k+1} \in \mathcal{A}[1] \quad (|a_i| = -1)$,

$$\begin{aligned}
&(\mu[1] \circ_G f)(a_1, \ldots, a_{k+1}) \\
&\quad = \mu[1](f \otimes id + id \otimes f)(a_1, \ldots, a_{k+1}) \\
&\quad = \mu[1](f(a_1, \ldots, a_k) \otimes a_{k+1}) + (-1)^{|a_1||f|} \mu[1](a_1 \otimes f(a_2, \ldots, a_{k+1}) \\
&\quad = (-1)^k a_1 f(a_2, \ldots, a_{k+1}) - f(a_1, \ldots, a_k) a_{k+1}
\end{aligned}$$

and

$$\begin{aligned}
(f \circ_G \mu[1])(a_1, \ldots, a_{k+1}) &= (f \circ \overline{\mu[1]})(a_1, \ldots, a_{k+1}) \\
&= \sum_{i=1}^{k} (-1)^i f(a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_{k+1}),
\end{aligned}$$

so $(\delta_H f)(a_1, \ldots, a_{k+1}) = \big((-1)^k \mu[1] \circ_G f + f \circ_G \mu[1]\big)(a_1, \ldots, a_{k+1})$ and

$$\delta_H f = f \circ_G \mu[1] - (-1)^{k-1} \mu[1] \circ_G f = [f, \mu[1]]_G = (-1)^{|f|} [\mu[1], f] = (-1)^{|f|} b(f),$$

that is, the Hochschild coboundary operator δ_H and b are equal up to the global sign $(-1)^{|f|}$. Therefore, we can work with b rather than δ_H.

The operator b is a graded derivation of $\mathcal{G}$. Indeed, for $f, g \in \mathcal{G}$ the graded Jacobi identity gives

$$\begin{aligned}
&(-1)^{|\mu[1]||g|}[\mu[1], [f, g]_G]_G + (-1)^{|g||f|}[g, [\mu[1], f]_G]_G + (-1)^{|f||\mu[1]|}[f, [g, \mu[1]]_G]_G = 0 \\
&\Leftrightarrow (-1)^{|g|} b([f, g]_G) + (-1)^{|g||f|}[g, b(f)]_G + (-1)^{|f|}[f, [g, \mu[1]]_G]_G = 0 \\
&\Leftrightarrow (-1)^{|g|} b([f, g]_G) - (-1)^{|g||f| + |g|(|f|+1)}[b(f), g]_G - (-1)^{|f| + |g||1|}[f, b(g)]_G = 0 \\
&\Leftrightarrow (-1)^{|g|} \Big(b([f, g]_G) - [b(f), g]_G - (-1)^{|f|}[f, b(g)]_G \Big) = 0 \\
&\Leftrightarrow b([f, g]_G) = [b(f), g]_G + (-1)^{|f|}[f, b(g)]_G.
\end{aligned} \tag{2.2.2}$$

To see that the square of operator b vanishes, taking $f = m$ in the previous equation gives

$$b^2(g) = [m, [m, g]_G]_G = [[m, m]_G, g]_G - [m, [m, g]_G]_G,$$

and since μ is associative, the first right-hand side term vanishes, so $b^2(g) = [m, [m, g]_G]_G = -[m, [m, g]_G]_G = 0$. This shows that the space $(\mathcal{G}, [\ ,\ ]_G, b)$ is a dg-Lie algebra.

The graded Jacobi identity implies that the Gerstenhaber bracket descends to a graded Lie bracket $[\ ,\ ]_s$ on the shifted cohomology space $\mathfrak{a}[1]$. This bracket is called the Schouten bracket, and we will denote the space $\mathfrak{a}[1]$ by $\mathfrak{g}$. So $(\mathfrak{g}, [\ ,\ ]_s)$ is a graded Lie algebra.

2.2.2 Sections

We shall call a *section* any graded linear injection ϕ of degree 0 of the Hochschild cohomology $\mathfrak{g}$ into the subspace of cocycles of $\mathcal{G}$, *i.e.*

$$\phi : \mathfrak{g} \to \mathcal{G} \text{ with } b \circ \phi = 0 \tag{2.2.3}$$

such that

$$p \circ \phi = id_{\mathfrak{g}} \tag{2.2.4}$$

where p denotes the canonical projection of the space of cocycles onto $\mathfrak{g}$. By elementary linear algebra this is always possible.

Proposition 2.2.2 *The set of all sections is in bijection with the set of all graded vectors space complements $\mathcal{H}$ of $B\mathcal{G}$ in $Z\mathcal{G}$.*

Proof. Let ϕ be a section. Define $\mathcal{H} := \mathrm{Im}\,\phi$. Since $b\phi = 0$, we have $\mathcal{H} \subset Z\mathcal{G}$. Let $f \in Z\mathcal{G}$. Then

$$p(f - \phi p f) = pf - p\phi p f \overset{(2.2.4)}{=} pf - pf = 0,$$

hence $f - \phi p f \in B\mathcal{G}$ and $Z\mathcal{G} = \mathcal{H} + B\mathcal{G}$. Let $g \in \mathcal{H} \cap B\mathcal{G}$. Then there exists $x \in \mathfrak{g}$ with $g = \phi x$ and $pg = 0$, hence $0 = pg = p\phi x = x$, and $g = 0$. Hence $\mathcal{H} \cap B\mathcal{G} = \{0\}$, and $Z\mathcal{G} = \mathcal{H} \oplus B\mathcal{G}$.

On the other hand, let $\mathcal{H}$ be a graded vector space complement to $B\mathcal{G}$ in $Z\mathcal{G}$, hence $Z\mathcal{G} = \mathcal{H} \oplus B\mathcal{G}$. Then the restriction of p to $\mathcal{H}$ is a linear bijection of degree 0 to $\mathfrak{g}$. Define $\tilde{\phi} : \mathfrak{g} \to \mathcal{H}$ by $\tilde{\phi} := \big(p|_{\mathcal{H}}\big)^{-1}$ and $\phi : \mathfrak{g} \to \mathcal{G}$ by the composition $\phi := i_{\mathcal{H}} \circ \tilde{\phi}$, where $i_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathcal{G}$ is the canonical injection. Clearly $b\phi = 0$ since the image of ϕ is in $\mathcal{H} \subset Z\mathcal{G}$. Moreover $p\phi = p \circ i_{\mathcal{H}} \circ \tilde{\phi} = p|_{\mathcal{H}}\tilde{\phi} = id_{\mathfrak{g}}$. Now let $\psi : \mathfrak{g} \to \mathcal{G}$ any other section having image $\mathcal{H}$, *i.e.* $\psi = i_{\mathcal{H}} \circ \tilde{\psi}$. Then $id_{\mathfrak{g}} = p\psi = p|_{\mathcal{H}}\tilde{\psi}$, and for all $h = \tilde{\psi}(x) \in \mathcal{H}$, $\tilde{\psi} p|_{\mathcal{H}} h = \tilde{\psi} p|_{\mathcal{H}} \tilde{\psi}(x) = \tilde{\psi}(x) = h$, hence $\tilde{\psi} = \big(p|_{\mathcal{H}}\big)^{-1} = \tilde{\phi}$, which implies $\psi = \phi$. □

2.2.3 Formality

In general a section $\phi : \mathfrak{g} \to \mathcal{G}$ will not be a morphism of graded Lie algebras $(\mathfrak{g}, [\ ,\ ]_s) \to (\mathcal{G}, [\ ,\ ]_G)$: by construction one only has for two classes $\xi, \eta \in \mathfrak{g}$

$$\phi[\xi,\eta]_s = [\phi(\xi), \phi(\eta)]_G + b\big(\phi_2(\xi,\eta)\big) \tag{2.2.5}$$

so ϕ is a morphism of graded Lie algebras up to a coboundary $b\big(\phi_2(\xi,\eta)\big)$. One may now hope that one can construct a sequence of linear maps $\phi_1 := \phi, \phi_2, \phi_3, \ldots, \phi_k, \ldots$ compatibles with the brackets up to higher homotopies. This gives the notion of L_∞-morphism, theses applications being seen as components of a graded coderivation of a graded symmetric algebra which links to the Definition 2.1.2.

We consider the shifted spaces $\mathfrak{g}[1]$ and $\mathcal{G}[1]$. We note $D = [\ ,\ ]_G[1]$ and $d = [\ ,\ ]_s[1]$ the shifted brackets on $\mathcal{G}$ and on $\mathfrak{g}$, we also have $b[1] = b$. Thanks to the shift it follows that both $[\ ,\ ]_G[1]$ and $[\ ,\ ]_s[1]$ are graded *symmetric* maps (Lemma 1.1.7), *i.e.* $[\ ,\ ]_G[1]$ is a degree 1 map from $\mathcal{S}^2(\mathcal{G}[1]) \to \mathcal{G}[1]$ and $[\ ,\ ]_s[1]$ is a degree 1 map from $\mathcal{S}^2(\mathfrak{g}[1]) \to \mathfrak{g}[1]$. Let $\overline{d} : \mathcal{S}(\mathfrak{g}[1]) \to \mathcal{S}(\mathfrak{g}[1])$ be the unique coderivation of $\mathcal{S}(\mathfrak{g}[1])$ induced by $[\ ,\ ]_s[1]$, and let $\overline{b+D}$ be the unique coderivation of $\mathcal{S}(\mathcal{G}[1])$ induced by $b + [\ ,\ ]_G[1]$.

Proposition 2.2.3 *The applications $\overline{d}$ and $\overline{b+D}$ are differentials, equipping respectively $(\mathfrak{g}, \overline{d})$ and $(\mathcal{G}, \overline{b+D})$ of structures of L_∞-algebras.*

Proof. Projecting on $\mathfrak{g}[1]$, we have

$$\overline{d} \circ \overline{d} = 0 \Leftrightarrow \mathrm{pr}_{\mathfrak{g}[1]} \overline{d} \circ \overline{d} = 0 \Leftrightarrow d \circ \overline{d} = 0 \Leftrightarrow d \circ_{NR} d = 0,$$

so for $\overline{d}$ to be a differential is equivalent that the square of d for the Nijenhuis-Richardson multiplication vanishes, which is equivalent by Lemma 1.1.7 to the graded Jacobi identity of the bracket $[\ ,\ ]_s$.

In the same way, projecting on $\mathcal{G}[1]$, we have

$$\overline{b+D} \circ \overline{b+D} = 0 \Leftrightarrow (b + D) \circ_{NR} (b + D) = 0.$$

As for d, the graded Jacobi identity of $[\ ,\ ]_G$ gives $D \circ_{NR} D = 0$, and since b only takes one argument, $b \circ_{NR} b = b \circ_G b = b \circ b = 0$ because we already know that b is a differential. It remains to prove that $b \circ_{NR} D + D \circ_{NR} b = 0$. For $f, g \in \mathcal{G}[1]$, on one hand we have $(b \circ_{NR} D)(f \bullet g) = b(D(f \bullet g)) = (-1)^{|f|} b([f, g]_G)$, and on the other hand,

$$\begin{aligned}(D \circ_{NR} b)(f \bullet g) &= D(b(f) \bullet g) + (-1)^{|f||g|} D(b(g) \bullet f) \\ &= (-1)^{|b(f)|}[b(f), g]_G + (-1)^{|f||g|+|b(g)|}[b(g), f]_G \\ &= (-1)^{|f|+1}[b(f), g]_G - (-1)^{|f||g|+|b(g)|+(|b(g)|+1)(|f|+1)}[f, b(g)]_G \\ &= (-1)^{|f|+1}[b(f), g]_G + [f, b(g)]_G.\end{aligned}$$

Therefore,

$$\begin{aligned}(b \circ_{NR} D + D \circ_{NR} b)(f \bullet g) &= (-1)^{|f|} \left(b([f, g]_G) - [b(f), g]_G - (-1)^{|f|+1}[f, b(g)]_G \right) \\ &= 0,\end{aligned}$$

since b is a graded derivation of $\mathcal{G}$ by (2.2.2). □

Definition 2.2.4 The Hochschild complex $\mathcal{G}$ associated to the associative algebra $(\mathcal{A}, \mu)$ is called *formal* if there is a L_∞-quasi-morphism (morphism of differential graded coalgebras of degree 0) $\Phi : \mathcal{S}(\mathfrak{g}[1]) \to \mathcal{S}(\mathcal{G}[1])$, *i.e.*

$$(\Phi \otimes \Phi) \circ \Delta_{\mathcal{S}\,\mathfrak{g}[1]} = \Delta_{\mathcal{S}\mathcal{G}[1]} \circ \Phi \qquad \text{and} \qquad \overline{b + D} \circ \Phi = \Phi \circ \overline{d}, \tag{2.2.6}$$

such that the restriction Φ_1 of Φ to $\mathfrak{g}[1]$ is a section. The map Φ is called a *formality map*.

The restrictions $\Phi_k := \mathrm{pr}_{\mathcal{G}[1]} \circ \Phi|_{\mathcal{S}^k\,\mathfrak{g}[1]}$ for any positive integer k are called the *Taylor coefficients of* Φ and determine Φ. They remedy order-by-order the above mentioned failure of the section map Φ_1 to be a morphism of graded Lie algebras.

In particular, a graded Lie algebra morphism $\phi : \mathfrak{g} \to \mathcal{G}$ defines a formality map $\Phi = \phi[1]$.

Originally in [Kon03], formality maps were defined from the space $\mathfrak{g}$ without shift by a family of multilinear applications. This definition is more useful for computations. Before giving it, we need to define the following signs.

Definition 2.2.5 For $k \in \mathbb{N}$, $\forall x_1, \ldots, x_{k+1} \in \mathfrak{g}$ (homogeneous), $\forall\ 1 \leqslant i < j \leqslant k+1$,

$$\epsilon_{ij}(x_1, \ldots, x_{k+1}) = (-1)^{\frac{k(k-1)}{2}} \cdot (-1)^{(|x_i|+|x_j|)(|x_1|+\cdots+|x_{i-1}|)+|x_j|(|x_{i+1}|+\cdots+|x_{j-1}|)}(-1)^{i+j},$$

and $\forall\ 1 \leqslant a \leqslant k$, $\forall\ 1 \leqslant i_1 < \cdots < i_k \leqslant k+1$,

$$\omega_a(x_1, \ldots, x_{k+1}) = \prod_{r=1}^{a} (-1)^{|x_{i_r}|(|x_1|+\cdots+\widehat{|x_{i_1}|}+\cdots+\widehat{|x_{i_{r-1}}|}+\cdots+|x_{i_r-1}|)} \cdot (-1)^{(k-a)(|x_{i_1}|+\cdots+|x_{i_a}|)}(-1)^{i_1+\cdots+i_a}(-1)^{\frac{(k+1-a)(k-a)}{2}}.$$

Proposition 2.2.6 *We consider a sequence of linear maps $\{\phi_k\}_{k\in\mathbb{N}^\star}$, with $\phi_k : \mathfrak{g}^{\otimes k} \to \mathcal{G}$, satisfying*

(i) ϕ_1 is a section

(ii) ϕ_k is of degree $1-k$

(iii) ϕ_k is graded antisymmetric, i.e. for all $1 \leqslant i \leqslant k$,

$$\phi_k(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_k) = -(-1)^{|x_i||x_{i+1}|}\phi_k(x_1, \ldots, x_k). \tag{2.2.7}$$

and such that for each $k \in \mathbb{N}$, $\forall x_1, \ldots, x_{k+1} \in \mathfrak{g}$ (homogeneous)

$$\sum_{1\leqslant i<j\leqslant k+1} \epsilon_{ij}\cdot \phi_k([x_i,x_j]_s,x_1,\cdots,\widehat{x_i},\cdots,\widehat{x_j},\cdots,x_{k+1})$$
$$= (-1)^{\frac{k(k+1)}{2}} b\phi_{k+1}(x_1,\ldots,x_{k+1}) \tag{2.2.8}$$
$$+\frac{1}{2}\sum_{a=1}^{k}\sum_{1\leqslant i_1<\cdots<i_a\leqslant k+1} \omega_a\cdot[\phi_a(x_{i_1},\ldots,x_{i_a}),\phi_{k+1-a}(x_1,\ldots,\widehat{x_{i_1}},\ldots,\widehat{x_{i_a}},\ldots,x_{k+1})]_G,$$

where $\epsilon_{ij} := \epsilon_{ij}(x_1,\ldots,x_{k+1})$ *et* $\omega_a := \omega_a(x_1,\ldots,x_{k+1})$.
Then $\Phi = e^{\tilde{\star}\varphi}$ *is a formality map, where* $\varphi = \sum_{k\geqslant 1}\phi_k[1]$.

Remark 2.2.7

Level $k=0$ We have $0 = b\phi_1$, which is equation (2.2.3) for sections.

Level $k=1$ The equation is

$$-\phi_1([x_1,x_2]_s) = -b\phi_2(x_1,x_2) + \frac{1}{2}\left(-[\phi_1(x_1),\phi_2(x_2)]_G + (-1)^{|x_1||x_2|}[\phi_1(x_2),\phi_1(x_1)]_G\right)$$

which is (2.2.5).

Level $k=2$ We obtain the formula

$$\begin{aligned}
&\phi_2([x_1,x_2]_s,x_3)\\
&+(-1)^{|x_3|(|x_1|+|x_2|)}\phi_2([x_3,x_1]_s,x_2)\\
&+(-1)^{(|x_2|+|x_3|)|x_1|}\phi_2([x_2,x_3]_s,x_1)\\
=&-b\phi_3(x_1,x_2,x_3)\\
&+(-1)^{|x_1|}[\phi_1(x_1),\phi_2(x_2,x_3)]_G\\
&+(-1)^{|x_2|}(-1)^{|x_1|(|x_2|+|x_3|)}[\phi_1(x_2),\phi_2(x_3,x_1)]_G\\
&+(-1)^{|x_3|}(-1)^{(|x_1|+|x_2|)|x_3|}[\phi_1(x_3),\phi_2(x_1,x_2)]_G.
\end{aligned}$$

which can be more easily written

$$\circlearrowleft_{x_1,x_2,x_3} \phi_2([x_1,x_2]_s,x_3) = -b\phi_3(x_1,x_2,x_3) + \circlearrowleft_{x_1,x_2,x_3} (-1)^{|x_1|}[\phi_1(x_1),\phi_2(x_2,x_3)]_G \tag{2.2.9}$$

with the convention that signs from the permutations of the x_i are comprised in the notation $\circlearrowleft_{x_1,x_2,x_3}$.

Sketch of the proof. The link with the first definition appears when taking the projection $\mathrm{pr}_{\mathcal{G}[1]} : \mathcal{S}(\mathfrak{g}[1]) \to \mathcal{G}[1]$ of the second equation of (2.2.6). With $d = d_2 := [\ ,\ ]_s[1]$ and $D = D_2 := [\ ,\ ]_G[1]$, we get

$$\varphi \circ (d_2 \tilde{\star} id_{\mathcal{S}(\mathfrak{g}[1])}) = (b + D_2) \circ e^{\tilde{\star}\varphi}.$$

Since $b = b \circ \mathrm{pr}_{\mathcal{G}[1]}$ and $D_2 = D_2 \circ \mathrm{pr}_{\mathcal{S}^2(\mathcal{G}[1])}$, this reads

$$\varphi \circ (d_2 \tilde{\star} id_{\mathcal{S}(\mathfrak{g}[1])}) = b \circ \varphi + \frac{1}{2} D_2 \circ \varphi \tilde{\star} \varphi.$$

Applying this to $y_1 \bullet \cdots \bullet y_{k+1} \in \mathcal{S}^{k+1}(\mathfrak{g}[1])$, we get

$$\begin{aligned}\sum_{1 \leqslant i < j \leqslant k+1} & (-1)^{(|y_i|+|y_j|)(|y_1|+\cdots+|y_{i-1}|)+|y_j|(|y_{i+1}|+\cdots+|y_{j-1}|)} \\ & \cdot \varphi_k(d_2(y_i, y_j) \bullet y_1 \bullet \cdots \bullet \widehat{y_i} \bullet \cdots \bullet \widehat{y_j} \bullet \cdots \bullet y_{k+1}) \\ = & b\varphi_{k+1}(y_1 \bullet \ldots \bullet y_{k+1}) \\ + \frac{1}{2} \sum_{a=1}^{k} \sum_{1 \leqslant i_1 < \cdots < i_a \leqslant k+1} & \prod_{r=1}^{a} (-1)^{|y_{i_r}|(|y_1|+\cdots+|\widehat{y_{i_1}}|+\cdots+|\widehat{y_{i_{r-1}}}|+\cdots+|y_{i_r-1}|)} \\ \cdot D_2(\varphi_a(y_{i_1} \bullet \ldots \bullet y_{i_a}) & \bullet \varphi_{k+1-a}(y_1 \bullet \ldots \bullet \widehat{y_{i_1}} \bullet \ldots \bullet \widehat{y_{i_a}} \bullet \ldots \bullet y_{k+1})]_G.\end{aligned}$$

With $\varphi_k := \phi_k[1] = s^{-1} \circ \phi_k \circ s^{\otimes k}$, $y_i := s^{-1} x_i$ and a careful computation of the signs, we obtain the equation (2.2.8). □

2.2.4 Application to deformation

The formal deformation theory of a dg-Lie algebra $\mathcal{G}$ associated to an associative algebra $(\mathcal{A}, \mu_0)$ is very simple if $\mathcal{G}$ is formal, as Kontsevich has shown in [Kon03]. (We follow here the presentation of [BM08, pp 321–322].) Let $\pi \in \mathrm{H_H}^2(\mathcal{A}, \mathcal{A})[[h]] = \mathfrak{a}^2[[h]] = \mathfrak{g}[1]^0[[h]]$. Suppose that

$$[\pi, \pi]_s = 0.$$

Then it is always possible to construct a formal associative deformation $\mu = \mu_0 + \mu_*$ where $\mu_* := \sum_{r=1}^{\infty} h^r \mu_r$ such that the cohomology class $[\mu_1]$ of μ_1 is equal to π.

Consider $\mathcal{S}(\mathfrak{g}[1])[[h]]$ and $\mathcal{S}(\mathcal{G}[1])[[h]]$ as topological bialgebras (with respect to the h-adic topology) with the canonical extension of all the structure maps. Note that the tensor product is no longer algebraic, but given by $\big(\mathcal{S}(\mathfrak{g}[1]) \otimes \mathcal{S}(\mathfrak{g}[1])\big)[[h]]$. Let $\bullet$ denote the shuffle-multiplication in a graded symmetric algebra. For a general graded vector space V it can be easily seen that the group-like elements of $\mathcal{S}\,V[[h]]$ are no longer exclusively given by $\mathbf{1}$, but by exponential functions of any primitive elements of degree zero, *i.e.* they take the form $e^{\bullet hv}$ with $v \in V^0[[h]]$. The image $\Phi(e^{\bullet h\pi})$ of the grouplike element $e^{\bullet h\pi}$ in $\mathcal{S}(\mathfrak{g}[1])[[h]]$ under the formality map Φ is a grouplike element in $\mathcal{S}(\mathcal{G}[1])[[h]]$ and thus takes the form $e^{\bullet \mu_*}$ with $\mu_* \in h\mathfrak{A}^2[[h]]$. Since $[\pi,\pi]_s = 0$ it follows that $d(e^{\bullet h\pi}) = 0$, and therefore $(b+D)(e^{\bullet h\mu_*}) = 0$. Projecting this last equation to $\mathcal{G}[1]^0[[h]] = \mathfrak{A}^2[[h]]$, we get the *Maurer-Cartan Equation*

$$0 = b\mu_* + \frac{1}{2}[\mu_*,\mu_*]_G = \frac{1}{2}[\mu_0+\mu_*,\mu_0+\mu_*]_G,$$

showing the associativity of $\mu = \mu_0 + \mu_*$. Hence $\mu := \mu_0 + \mu_*$ is a formal associative deformation of the algebra $(\mathcal{A},\mu_0)$.

2.3 Case of Lie algebras

2.3.1 Polyvectors fields and polynomials functions

We consider polyvectors fields as polynomials functions, we reproduce here the explanation of [BM08, pp. 324–325] of this correspondance.

Let E be a finite dimensional $\mathbb{K}$-vector space which is ungraded ($\mathbb{Z}$-graded of degree 0), and $\mathcal{A} := \mathcal{S}E$ its symmetric algebra. According to the Cartan-Eilenberg Theorem 1.2.1, for each $n \in \mathbb{N}$,

$$\mathrm{H_H}^n(\mathcal{A},\mathcal{A}) \cong \bigwedge^n E^* \otimes \mathcal{S}E =: \mathcal{T}^n_{poly}$$

where the latter space is the space of algebraic polyvector fields of rank n. We set $\mathcal{T}_{poly} = \bigoplus_{n=0}^{\infty} \mathcal{T}^n_{poly}$.

For the computations, we shall use the canonical identification of $\mathcal{S}E$ with the algebra of all polynomial functions on the dual space E^*. Let $\{e_i\}_{i=1}^n$ be a basis of E, and let $\{e^i\}_{i=1}^n$ be the dual basis. We can regard any $f \in \mathcal{S}E$ as a polynomial in the coordinates x_i (upon writing each $x \in E^*$ as $x = \sum_{i=1}^n x_i e^i$). The polyvector fields are now polynomial functions with values in $\bigwedge E^*$. Using partial derivatives $\partial_i := \partial/\partial x_i$ in $\mathcal{S}E$ and interior products $\imath_{e_i}$ with respect to the dual basis in $\bigwedge E^*$, we can write the projected Gerstenhaber bracket $[\ ,\ ]_s$ in its classical form as an algebraic Schouten bracket

$$[\xi,\eta]_s := (-1)^{|\xi|-1}\sum_{i=1}^n \imath_{e_i}\xi \wedge \partial_i\eta - (-1)^{|\eta|-1}\sum_{i=1}^n \imath_{e_i}\eta \wedge \partial_i\xi, \qquad (2.3.1)$$

where $|\xi|$ et $|\eta|$ are the ranks of the polyvectors fields ξ and η in $\mathcal{T}_{poly}$.

As the Gerstenhaber bracket, the Schouten bracket defines a graded Lie bracket on the shifted space $\mathcal{T}_{poly}[1]$. Therefore we have the identities from the Definition 1.1.1: with $\xi,\eta,\zeta \in \mathcal{T}_{poly}[1]$ we have

$$[\eta,\xi]_s = -(-1)^{|\eta||\xi|}[\xi,\eta]_s,$$
$$(-1)^{|\xi||\zeta|}[[\xi,\eta]_s,\zeta]_s + (-1)^{|\eta||\xi|}[[\eta,\zeta]_s,\xi]_s + (-1)^{|\zeta||\eta|}[[\zeta,\xi]_s,\eta]_s = 0,$$

degrees being taken in $\mathcal{T}_{poly}[1]$.

On the other hand there is the pointwise exterior multiplication $\wedge$ which makes $\mathcal{T}_{poly} = \mathcal{S}(E^* \oplus E^*[-1])$ a graded commutative algebra. The Schouten bracket and the exterior multiplication are compatible by the graded Leibniz rule

$$[\xi,\eta\wedge\zeta]_s = [\xi,\eta]_s \wedge \zeta + (-1)^{|\xi|(|\eta|+1)}\eta\wedge[\xi,\zeta]_s, \tag{2.3.2}$$

(if η is of degree $|\eta|$ in $\mathcal{T}_{poly}[1]$ then it is of degree $|\eta|+1$ in $\mathcal{T}_{poly}$).

Throughout the computations, we will use the fact that $\mathcal{T}_{poly}$ is an $\mathcal{S}E$-module *i.e.* for $\xi \in \mathcal{T}_{poly}$, $f \in \mathcal{S}E$, $f \wedge \xi = f\xi$. We also give some examples of Schouten brackets which will be used later.

Example 2.3.1 The space $\mathcal{T}^1_{poly}$ is the space of vectors fields, *i.e.* the derivations of $\mathcal{S}E = \mathcal{T}^0_{poly}$. For $f,g \in \mathcal{S}E$, $X,Y \in \mathcal{T}^1_{poly}$

$$[f,g]_s = 0,$$
$$[X,f]_s = X(f) = \sum_{i=1}^n X_i\partial_i f,$$
$$[X,Y]_s = [X,Y] = \sum_{i=1}^n X_i\partial_i Y - \sum_{i=1}^n Y_i\partial_i X,$$

with in particular

$$[f\partial_i, g\partial_j] = f(\partial_i g)\partial_j - g(\partial_j f)\partial_i.$$

2.3.2 Linear Poisson structure

Let $E = (\mathfrak{g},[\ ,\ ])$ be a finite n-dimensional Lie algebra and $\mathfrak{g}^*$ its algebraic dual. The Lie algebra structure $[\ ,\ ]$ of $\mathfrak{g}$ can be used to define a Poisson bracket on $\mathfrak{g}^*$: for $f,g \in \mathcal{S}\mathfrak{g}$ and $x \in \mathfrak{g}^*$

$$\{f,g\}(x) := x\big([df(x),dg(x)]\big), \tag{2.3.3}$$

which also writes

$$\{f,g\}(x) = P(df,dg)(x) = \frac{1}{2}\sum_{1\leqslant i,j,k\leqslant n} C^k_{ij}x_k\partial_i f\wedge\partial_j g$$

with $P \in \mathcal{S}^1\,\mathfrak{g} \otimes \bigwedge^2 \mathfrak{g}^* = \mathcal{T}^2_{poly}$ the bivector defined by

$$P = \frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} P^{ij} \partial_i \wedge \partial_j \qquad \text{where} \qquad P^{ij}(x) = \sum_{k=1}^{n} C^k_{ij} x_k \tag{2.3.4}$$

and C^k_{ij} the structure constants of $\mathfrak{g}$.

Proposition 2.3.2 *With the preceding notations, $(\mathcal{S}\,\mathfrak{g}, \cdot, \{\ ,\ \})$ is a Poisson algebra.*

Proof. The bracket defined by P satisfies the Leibniz identity because of the Leibniz rule for derivations. The Jacobi relation for P is obtained recursively, using the fact that $\{e_i, e_j\} = [e_i, e_j]$ and that the bracket $[\ ,\ ]$ satisfies the Jacobi identity by definition. □

2.3.3 Formality

The fact that P is a Poisson structure is equivalent to $[P,P]_s = 0$. Thus, the operator defined by $\delta_P := [P,\ ]_s$ is a differential on the complex $\mathcal{T}_{poly}$, whose cohomology is called *Poisson cohomology*.

By computation, the complex $(\mathcal{T}_{poly}, \delta_P)$ identifies with the Chevalley-Eilenberg complex $(\mathrm{C}_{\mathrm{CE}}(\mathfrak{g}, \mathcal{S}\,\mathfrak{g}), \delta_{CE})$. We note $\mathfrak{A}[1] := \mathcal{T}_{poly}[1]$ this differential graded Lie algebra and $\mathfrak{a}[1] := \mathrm{H}_{\mathrm{CE}}(\mathfrak{g}, \mathcal{S}\,\mathfrak{g})[1]$ its cohomology with values in $\mathcal{S}\,\mathfrak{g}$, which is again a graded Lie algebra, with the induced Schouten bracket $[\ ,\ ]'_s$.

As in the case (Section 2.2.3) where the initial algebra is associative, we consider $\overline{d}$ the unique coderivation of $\mathcal{S}(\mathfrak{a}[2])$ induced by $[\ ,\ ]'_s[1]$ and $\overline{\delta_{CE} + D}$, the unique coderivation of $\mathcal{S}(\mathfrak{A}[2])$ induced by $\delta_{CE} + [\ ,\ ]_s[1]$. They are also differentials, which rewrites as follow.

Proposition 2.3.3 *The applications $\overline{d}$ and $\overline{\delta_{CE} + D}$ are differentials, equipping respectively $(\mathfrak{a}[1], \overline{d})$ and $(\mathfrak{A}[1], \overline{\delta_{CE} + D})$ of structures of L_∞-algebras.*

Definition 2.3.4 The Chevalley-Eilenberg complex $\mathfrak{A}[1]$ associated to the Lie algebra $(\mathfrak{g}, [\ ,\ ])$ is said to be *formal* if there is a L_∞-quasi-isomorphism $\Phi : \mathcal{S}(\mathfrak{a}[2]) \to \mathcal{S}(\mathfrak{A}[2])$, *i.e.*

$$(\Phi \otimes \Phi) \circ \Delta_{\mathcal{S}\,\mathfrak{a}[2]} = \Delta_{\mathcal{S}\,\mathfrak{A}[2]} \circ \Phi \qquad \text{et} \qquad \overline{\delta_{CE} + D} \circ \Phi = \Phi \circ \overline{d}, \tag{2.3.5}$$

such that the restriction Φ_1 of Φ to $\mathfrak{a}[2]$ is a section. The map Φ is called a *formality map*.

As we will see, even if there is no formality map between $\mathcal{S}(\mathfrak{a}[2])$ and $\mathcal{S}(\mathfrak{A}[2])$, we can always modify the L_∞ structure induced by

$d = d_2 = [\ ,\ ]'_s$ in $d = \sum_{n\geqslant 2} d_n$ and by induction build the components of the differential $\overline{d}$ and of the L_∞-morphism $\Phi = \overline{\varphi}$. They satisfy the equation

$$\Phi \circ \overline{d} = \overline{\delta_{CE} + D} \circ \Phi, \tag{2.3.6}$$

which writes, after projection on $\mathfrak{A}[2]$ and using that $D = D_2 = [\ ,\ ]_s[1]$,

$$\varphi \circ (d \tilde{\star} id_{\mathcal{S}(\mathfrak{a}[2])}) = \delta_{CE} \circ \varphi + \frac{1}{2} D_2 \circ \varphi \tilde{\star} \varphi. \tag{2.3.7}$$

Since $d = \sum_{n\geqslant 2} d_n$, evaluating on $y_1 \bullet \cdots \bullet y_{k+1} \in \mathcal{S}(\mathfrak{a}[2])$ gives

$$\begin{aligned} &\sum_{a=2}^{k+1} \sum_{1\leqslant i_1<\cdots<i_a\leqslant k+1} \nu_a(y_1,\ldots,y_{k+1}) \\ &\quad \cdot \varphi_{k+2-a}(d_a(y_{i_1} \bullet \ldots \bullet y_{i_a}) \bullet y_1 \bullet \ldots \bullet \widehat{y_{i_1}} \bullet \ldots \bullet \widehat{y_{i_a}} \bullet \ldots \bullet y_{k+1}) \\ &= \delta_{CE} \varphi_{k+1}(y_1 \bullet \ldots \bullet y_{k+1}) \\ &+ \frac{1}{2} \sum_{a=1}^{k} \sum_{1\leqslant i_1<\cdots<i_a\leqslant k+1} \nu_a(y_1,\ldots,y_{k+1}) \\ &\quad \cdot D_2(\varphi_a(y_{i_1} \bullet \ldots \bullet y_{i_a}) \bullet \varphi_{k+1-a}(y_1 \bullet \ldots \bullet \widehat{y_{i_1}} \bullet \ldots \bullet \widehat{y_{i_a}} \bullet \ldots \bullet y_{k+1}). \end{aligned} \tag{2.3.8}$$

Compared to the original definition Proposition 2.2.6, the maps $\varphi_k = \phi_k[1]$ are of degree 0 ($|\phi_k[1]| = 1(k-1) + |\phi_k| = 0$), and the signs

$$\nu_a(y_1,\ldots,y_{k+1}) = \prod_{r=1}^{a} (-1)^{|y_{i_r}|(|y_1|+\cdots+\widehat{|y_{i_1}|}+\cdots+\widehat{|y_{i_{r-1}}|}+\cdots+|y_{i_r-1}|)}$$

for $1 \leqslant a \leqslant k+1$ only come from the permutation which place at the beginning the elements of indices $i_1,\ldots,i_a$.
However, care of signs must be taken when working with the shifted maps, for example $D_2(x,y) = (-1)^{|x|}[x,y]_s$.

2.4 Universal enveloping algebras

Let $(\mathfrak{g}, [\ ,\ ])$ be a finite-dimensional Lie algebra and $\mathcal{S}\,\mathfrak{g}$ its associated Poisson algebra. With the previous notations, we consider the chain complexes below and their associated shifted brackets.

$\mathfrak{a} := \mathrm{H_H}(\mathcal{S}\,\mathfrak{g}, \mathcal{S}\,\mathfrak{g})$	$\mathfrak{a}' := \mathrm{H_{CE}}(\mathfrak{g}, \mathcal{S}\,\mathfrak{g})$	$d := [\ ,\ ]_s[1]$	$d' := [\ ,\ ]'_s[1]$
$\mathfrak{A} := \mathrm{C_H}(\mathcal{S}\,\mathfrak{g}, \mathcal{S}\,\mathfrak{g})$	$\mathfrak{A}' := \mathrm{C_H}(\mathcal{U}\,\mathfrak{g}, \mathcal{U}\,\mathfrak{g})$	$D := [\ ,\ ]_G[1]$	$\mathbf{D} := [\ ,\ ]_{\mathbf{G}}[1]$

The spaces $(\mathfrak{a}[1],[\ ,\ ]_s)$ and $(\mathfrak{a}'[1],[\ ,\ ]'_s)$ are graded Lie algebras and $(\mathfrak{A}[1],[\ ,\ ]_G,b)$ and $(\mathfrak{A}'[1],[\ ,\ ]_{\mathbf{G}},\mathbf{b})$ are differential graded Lie algebras. The bracket $[\ ,\ ]_{\mathbf{G}}$ and the differential $\mathbf{b} := [\mathbf{m},\ ]_{\mathbf{G}}$ are in boldface to remind that they come from the associative algebra $(\mathcal{U}\,\mathfrak{g},\mathbf{m})$, whose multiplication is different from those of $(\mathcal{S}\,\mathfrak{g},\cdot)$.

In his celebrated article [Kon03], Kontsevich gives an explicit formula for a formality map

$$\Phi : (\mathcal{S}(\mathfrak{a}[2]),\overline{d}) \to (\mathcal{S}(\mathfrak{A}[2]),\overline{b+D}).$$

Considering $\mathcal{U}\,\mathfrak{g}$ as a convergent deformation of $\mathcal{S}\,\mathfrak{g}$, Bordemann et Makhlouf ([BM08][Theorem 6.1]) twist by conjugation this map and obtain a L_∞-morphism

$$\Phi' : (\mathcal{S}(\mathfrak{a}[2]),\overline{\delta_{CE}+d}) \to (\mathcal{S}(\mathfrak{A}'[2]),\overline{\mathbf{b}+\mathbf{D}}).$$

We use here the identification between the chain complexes $(\mathfrak{a} \cong \mathcal{T}_{poly},\delta_P)$ and $(\mathrm{C}_{\mathrm{CE}}(\mathfrak{g},\mathcal{S}\,\mathfrak{g}),\delta_{CE})$.

This enable to show that the formality associated to a Lie algebra $\mathfrak{g}$ is equivalent to the one for its universal enveloping algebra $\mathcal{U}\,\mathfrak{g}$.

Theorem 2.4.1 ([BM08]) *The formality of the chain complex* $\mathrm{C}_{\mathrm{CE}}(\mathfrak{g},\mathcal{S}\,\mathfrak{g})$ *is equivalent to the formality of the chain complex* $\mathrm{C}_{\mathrm{H}}(\mathcal{U}\,\mathfrak{g},\mathcal{U}\,\mathfrak{g})$.

Proof. Suppose that the chain complex $\mathrm{C}_{\mathrm{CE}}(\mathfrak{g},\mathcal{S}\,\mathfrak{g})$ is formal, so there exists a L_∞-quasi-isomorphism

$$\Phi'' : (\mathcal{S}(\mathfrak{a}'[2]),\overline{d'}) \to (\mathcal{S}(\mathfrak{a}[2]),\overline{\delta_{CE}+d})$$

such that Φ''_1 is a section. Composing with Φ', we obtain a formality map

$$\mathit{\Phi} : (\mathcal{S}(\mathfrak{a}'[2]),\overline{d'}) \to (\mathcal{S}(\mathfrak{A}'[2]),\overline{\mathbf{b}+\mathbf{D}}).$$

Since $\mathfrak{a}' = \mathrm{H}_{\mathrm{CE}}(\mathfrak{g},\mathcal{S}\,\mathfrak{g}) \cong \mathrm{H}_{\mathrm{H}}(\mathcal{U}\,\mathfrak{g},\mathcal{U}\,\mathfrak{g})$ by the Cartan-Eilenberg Theorem 1.2.1, $\mathit{\Phi} = \Phi' \circ \Phi''$ indeed is a formality map between the symmetric coalgebras constructed on the Hochschild cohomology and chain complex of $\mathcal{U}\,\mathfrak{g}$.

Conversely, if the chain complex $\mathrm{C}_{\mathrm{H}}(\mathcal{U}\,\mathfrak{g},\mathcal{U}\,\mathfrak{g})$ is formal, we have the previous map $\mathit{\Phi}$ and we obtain the formality map Φ'' by composing with the inverse L_∞-morphism Φ'^{-1}. □

2.5 Perturbation Lemma

Let $(\mathcal{G},\overline{D})$ and $(\mathfrak{g},\overline{d})$ be two L_∞-algebras. We don't necessarily have a L_∞-morphism from $(\mathfrak{g},\overline{d})$ to $(\mathcal{G},\overline{D})$. Nevertheless, we may perturb the map $\overline{d}$ in $\overline{d'}$ and construct a map $\phi : \mathcal{S}\,(\mathfrak{g}[1]) \to \mathcal{G}[1]$ such that

$\Phi = \overline{\phi} : \mathcal{S}(\mathfrak{g}[1]) \to \mathcal{S}(\mathcal{G}[1])$ is a L_∞-morphism between $(\mathfrak{g}, \overline{d'})$ and $(\mathcal{G}, \overline{D})$.

The *Perturbation Lemma* is a result coming from homological perturbation theory which, beginning with a contraction between two chain complexes and a perturbation of one of the differentials, enable to build a new contraction between the chain complexes endowed with perturbed differentials. In [Hue11, Hue10], Huebschmann deals with the case of dg-Lie and L_∞ algebras ; he shows how to construct a new contraction natural in the given structures. This allows to inductively build a formality map. This construction was also done by Bordemann et al. in [BGH⁺05, Bor] for L_∞ structure and for other operads.

More precisely, we have the following results. We first define the terms *contraction* and *perturbation*, then we state the *Perturbation Lemma*. We then proceed to the construction of a L_∞-morphism in the case where cohomology is concentrated only in degrees 0 and 1, following [BGH⁺05, A.4, Prop. A.3]. Finally, we give the result without particular hypothesis on the cohomology.

Definition 2.5.1 A *contraction* consists of two chain complexes (U, d_U) and (V, d_V) together with chain maps $i : U \to V$, $p : V \to U$, *i.e.*

$$d_V \circ i = i \circ d_U, \qquad d_U \circ p = p \circ d_V \tag{2.5.1a}$$

and a map $h : V \to V$ of degree -1 such that

$$p \circ i = id_U \tag{2.5.1b}$$

$$id_V - i \circ p = d_V h + h d_V \tag{2.5.1c}$$

$$h^2 = 0 \qquad h \circ i = 0 \qquad p \circ h = 0. \tag{2.5.1d}$$

Then p is a surjection called the *projection*, i is an injection called the *inclusion* and h is an *homotopy* between id_V and $i \circ p$. We sum up equations (2.5.1) with the diagram

$$(U, d_U) \underset{p}{\overset{i}{\rightleftarrows}} (V, d_V) \quad \circlearrowleft h\,. \tag{2.5.2}$$

Remark 2.5.2

1. Condition (2.5.1c) implies that the cohomologies of U and V are isomorphic. Denoting by $[f, g] := f \circ g - (-1)^{|f||g|} g \circ f$ the graded commutator of two maps, this equation (2.5.1c) also rewrites $id_V - i \circ p = [d_V, h]$.

2. Equations (2.5.1d) are called *side conditions*. They are equivalent to the equation $hd_V h = h$ and can always be satisfied replacing h by $hd_V h$ if needed.

Definition 2.5.3 A *perturbation* of the differential d_V is a morphism $\delta : V \to V$ of degree +1 such that $(d_V + \delta)^2 = 0 \Leftrightarrow \delta^2 + [\delta, d_V] = 0$.

Lemma 2.5.4 (Perturbation Lemma) *Given a filtered contraction*

$$(U, d_U) \underset{p}{\overset{i}{\rightleftarrows}} (V, d_V) \qquad \circlearrowleft h\,.$$

and a perturbation δ of d_V, let

$$\tilde{\imath} = (id_V + h\delta)^{-1} i = \sum_{n\in\mathbb{N}} (-h\delta)^n i \qquad \tilde{h} = (id_V + h\delta)^{-1} h = \sum_{n\in\mathbb{N}} (-h\delta)^n h$$
$$\tilde{p} = p(id_V + \delta h)^{-1} = \sum_{n\in\mathbb{N}} p(-\delta h)^n \qquad \tilde{\delta} = p(id_V + \delta h)^{-1}\delta i = \sum_{n\in\mathbb{N}} p(-\delta h)^n \delta i.$$

If the filtrations on U and V are complete, then the above series converge, $\tilde{\delta}$ is a perturbation of d_U and there exists a filtered contraction

$$(U, d_U + \tilde{\delta}) \underset{\tilde{p}}{\overset{\tilde{\imath}}{\rightleftarrows}} (V, d_V + \delta) \qquad \circlearrowleft \tilde{h}\,.$$

This result was first published in [Bro65].

Proof. We show that the new maps $\tilde{\imath}, \tilde{p}, \tilde{h}, d_V + \delta, d_U + \tilde{\delta}$ satisfies equations (2.5.1). Setting $id := id_V$, we remark first that

$$\delta(id + h\delta)^{-1} = \sum_{n\in\mathbb{N}} \delta(-h\delta)^n = \sum_{n\in\mathbb{N}} (-\delta h)^n \delta = (id + \delta h)^{-1}\delta,$$
$$(id + h\delta)^{-1} h = \sum_{n\in\mathbb{N}} (-h\delta)^n h = \sum_{n\in\mathbb{N}} h(-\delta h)^n = h(id + \delta h)^{-1};$$
$$d_V(id + h\delta) + (id - [d_V, h])\delta = d_V + \delta - hd_V\delta = (id + h\delta)(d_V + \delta),$$
$$(id + \delta h)d_V + \delta(id - [d_V, h]) = d_V + \delta - \delta d_V h = (d_V + \delta)(id + \delta h),$$

using $\delta^2 = -\delta d_V - d_V\delta$ in the last two equations. So we obtain

$$\begin{aligned}
\tilde{\imath} \circ (d_U + \tilde{\delta}) &= (id + h\delta)^{-1}(i \circ d_U + i \circ p \circ \delta(id + h\delta)^{-1} i) \\
&= (id + h\delta)^{-1}(d_V \circ i + (id - [d_V, h])\delta\tilde{\imath}) \\
&= (id + h\delta)^{-1}\Big(d_V(id + h\delta) + (id - [d_V, h])\delta\Big)\tilde{\imath} \\
&= (id + h\delta)^{-1}\Big((id + h\delta)(d_V + \delta)\Big)\tilde{\imath} \\
&= (d_V + \delta)\tilde{\imath}
\end{aligned}$$

and similarly

$$\begin{aligned}(d_U+\tilde{\delta})\circ\tilde{p} &= (d_U\circ p+p(id+\delta h)^{-1}\delta\circ i\circ p)(id+\delta h)^{-1}\\ &= (p\circ d_V+\tilde{p}\delta(id-[d_V,h]))(id+\delta h)^{-1}\\ &= \tilde{p}\Big((id+\delta h)d_V+\delta(id-[d_V,h])\Big)(id+\delta h)^{-1}\\ &= \tilde{p}\Big((d_V+\delta)(id+\delta h)\Big)(id+\delta h)^{-1}\\ &= \tilde{p}(d_V+\delta).\end{aligned}$$

Moreover

$$\begin{aligned}id-\tilde{\imath}\circ\tilde{p} &= id-(id+h\delta)^{-1}i\circ p(id+\delta h)^{-1}\\ &= (id+h\delta)^{-1}\Big((id+h\delta)(id+\delta h)-(id-[d_V,h])\Big)(id+\delta h)^{-1}\\ &= (id+h\delta)^{-1}\Big([h,d_V+\delta]+h\delta^2 h\Big)(id+\delta h)^{-1}\end{aligned}$$

and

$$\begin{aligned}[(d_V+\delta),\tilde{h}] &= (id+h\delta)^{-1}h(d_V+\delta)+(d_V+\delta)h(id+\delta h)^{-1}\\ &= (id+h\delta)^{-1}\Big(h(d_V+\delta)(id+\delta h)+(id+h\delta)(d_V+\delta)h\Big)(id+\delta h)^{-1}\\ &= (id+h\delta)^{-1}\Big([h,d_V+\delta]-h[\delta,d_V]h\Big)(id+\delta h)^{-1}\end{aligned}$$

thus we have

$$id-\tilde{\imath}\circ\tilde{p}=[(d_V+\delta),\tilde{h}].$$

Finally,

$$\begin{aligned}\tilde{p}\circ\tilde{\imath} &= p\sum_{n\in\mathbb{N}}(-\delta h)^n\sum_{m\in\mathbb{N}}(-h\delta)^m i\\ &= p\sum_{n+m=p}(-\delta h)^n(-h\delta)^m i=id_U\end{aligned}$$

because of side conditions (2.5.1d). These conditions on h, i, p implies the same conditions on $\tilde{h}, \tilde{\imath}, \tilde{p}$.

$$\begin{aligned}\tilde{h}^2 &= (id+h\delta)^{-1}h\circ h(id+\delta h)^{-1}=0\\ \tilde{h}\circ\tilde{\imath} &= (id+h\delta)^{-1}h\circ(id+h\delta)^{-1}i=(id+h\delta)^{-1}\circ h\circ i=0\\ \tilde{p}\circ\tilde{h} &= p(id+\delta h)^{-1}\circ h(id+\delta h)^{-1}=p\circ h\circ(id+\delta h)^{-1}=0\end{aligned}$$

□

2.5.1 Two-degree cohomology

Let $(\mathcal{A},\mu)$ be an associative algebra, $\mathfrak{A}=\bigoplus_{n=0}^{\infty}\mathfrak{A}^n$ the associated space of Hochschild cochains and $\mathfrak{a}$ its cohomology.

We suppose that the cohomology is concentred only on two degrees $\mathfrak{a} = \mathfrak{a}^0 \oplus \mathfrak{a}^1$. From the shifted point of view $\mathcal{G} := \mathfrak{A}[1]$ and $\mathfrak{g} := \mathfrak{a}[1]$, in order to work with Lie algebras, this writes $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$.

We use the previous notations $\overline{d}$ and $\overline{b+D}$ from Section 2.2 for the differentials induced by the shifted brackets, and we construct order by order the perturbed L_∞ structure $\overline{d'}$ on $\mathfrak{g}$ to obtain a L_∞-morphism between $(\mathfrak{g}, \overline{d'})$ and $(\mathcal{G}, \overline{b+D})$.

First, the following lemma limits the 'length' of ϕ.

Lemma 2.5.5 *On $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$, a map $\phi_k : \mathfrak{g} \to \mathcal{G}$ of degree $1-k$ vanishes for $k \geqslant 3$.*

Proof. Beginning with $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$, since ϕ_1 is a map of degree 0 we have that

$$\phi_1(\mathfrak{g}^{-1}) \subset \mathcal{G}^{-1} = \mathfrak{A}^0 \quad \text{and} \quad \phi_1(\mathfrak{g}^0) \subset \mathcal{G}^0 = \mathfrak{A}^1.$$

In the same way, ϕ_2 is a map of degree -1 hence

$$\begin{aligned} \phi_2(\mathfrak{g}^{-1}, \mathfrak{g}^{-1}) &\subset \mathcal{G}^{-3} = \mathfrak{A}^{-2} = \{0\} \\ \phi_2(\mathfrak{g}^{-1}, \mathfrak{g}^{0}) &\subset \mathcal{G}^{-2} = \mathfrak{A}^{-1} = \{0\} \\ \phi_2(\mathfrak{g}^{0}, \mathfrak{g}^{0}) &\subset \mathcal{G}^{-1} = \mathfrak{A}^{0}. \end{aligned}$$

For all $k \in \mathbb{N}$, ϕ_k is a map of degree $1-k$ hence

$$\phi_k(\mathfrak{g}^{i_1}, \dots, \mathfrak{g}^{i_k}) \subset \mathcal{G}^{i_1 + \dots + i_k + 1 - k}$$

and since $-1 \leqslant i_1, \dots, i_k \leqslant 0$, we have $i_1 + \dots + i_k + 1 - k \leqslant 1 - k \leqslant -2$ for $k \geqslant 3$ so $\phi_k = 0$ for $k \geqslant 3$. □

So the map ϕ is of the form $\phi = \phi_1 + \phi_2$.

Theorem 2.5.6 *There is a L_∞-structure $\overline{d'} = \overline{d_2 + d_3}$ which confers $(\mathfrak{g}, \overline{d'})$ the structure of a L_∞-algebra, and a L_∞-morphism $\overline{\phi} : \mathcal{S}(\mathfrak{g}[1]) \to \mathcal{S}(\mathcal{G}[1])$ from $(\mathfrak{g}, \overline{d'})$ to $(\mathcal{G}, \overline{b+D})$, i.e. which satisfies $\overline{b+D}\ \overline{\phi} = \overline{\phi}\ \overline{d'}$. Moreover, the restriction of $d_3 : \mathcal{S}^3(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ to $\mathcal{S}^3(\mathfrak{g}[1]^{-1})$ is a 3-cocycle of the Chevalley-Eilenberg cohomology of $\mathfrak{g}^0$ with values in $\mathfrak{g}^{-1}$.*

Proof. We note $\overline{E} := \overline{b+D}\ \overline{\phi} - \overline{\phi}\ \overline{d'}$ and $\overline{F} = \overline{d'}^2$. Since $|\overline{b+D}| = 1$ and $|\overline{\phi}| = 0$, we have $|d'| = 1$ and $d' = d_2 + d_3$. Since $\overline{b+D}$ and $\overline{d'}$ are graded Lie coderivations and $\overline{\phi}$ is a morphism of Lie coalgebra, $\overline{E}$ is a Lie coderivation along $\overline{\phi}$, so $\overline{E} = \overline{\phi} \tilde{\star} E$. Moreover, $\overline{b+D}^2 = 0$ implies that

$$\overline{b+D}\ \overline{E} + \overline{E}\ \overline{d'} + \overline{\phi}\ \overline{F} = 0. \tag{2.5.3}$$

We want to construct ϕ and d' such that $\overline{E}$ and $\overline{F}$ vanish. Projecting on the ground spaces, we have

$$\overline{E} = 0 \Leftrightarrow E := \mathrm{pr}_{\mathcal{G}[1]}\overline{E} = 0, \qquad \overline{F} = 0 \Leftrightarrow F := \mathrm{pr}_{\mathfrak{g}[1]}\overline{F} = 0. \tag{2.5.4}$$

Since $d' : \mathcal{S}^{\geqslant 2}\mathfrak{g}[1] \to \mathcal{G}[1]$, F takes at least three arguments so we have $F_1 := F|_{\mathcal{S}^1\mathfrak{g}[1]} = 0$ and $F_2 := F|_{\mathcal{S}^2\mathfrak{g}[1]} = 0$.

We project the equation (2.5.3) on $\mathcal{G}[1]$ and then evaluate it on n arguments.

$n = 1$ Since $D = D_2 : \mathcal{S}^2(\mathcal{G}[1]) \to \mathcal{G}[1]$ and $d = d_2 : \mathcal{S}^2(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ take two arguments, the only remaining term is $bE_1 = 0$, with $E_1 = b\phi_1$.
Hence $\phi_1(\mathfrak{g}^{-1}) \subset Z\mathfrak{A}^0$ and $\phi_1(\mathfrak{g}^0) \subset Z\mathfrak{A}^1 = \mathrm{Der}(\mathcal{A})$, with $\mathfrak{A}^0 = \mathcal{A}$.
We choose a linear injection $\phi_1 : \mathfrak{g} \to Z\mathcal{G}$ such that $p\phi_1 = id_{\mathfrak{g}}$, so $E_1 = b\phi_1 = 0$.

$n = 2$ We have $b\overline{E}_2 + D\overline{E}_2 + Ed_2 + \phi_1\cancel{F_2} = 0$. As $E_1 = 0$, $\overline{E}_2 = E_2$ and since $D = D_2$ takes two arguments and E_2 only gives one, and since d_2 gives one argument to E and $E_1 = 0$, it only remains $bE_2 = 0$. The term E_2 writes $E_2 = b\phi_2 + \cancel{b\overline{\phi_1}|_2} + D\overline{\phi_1}|_2 - \phi_1 d_2$. By definition of D_2, d_2 and the projection $p : Z\mathcal{G} \to \mathfrak{g}$, $p(D\overline{\phi_1}|_2 - \phi_1 d_2) = pD\overline{\phi_1}|_2 - d_2 = 0$ so $D\overline{\phi_1}|_2 - \phi_1 d_2$ is a coboundary. We can then choose ϕ_2 such that $E_2 = 0$. The map $\phi_2 : \mathcal{S}^2(\mathfrak{g}^0) \to \mathcal{G}^{-1} = \mathcal{A}$ is defined up to a coboundary, if another map ϕ_2' is taken, we have $\mathrm{Ker}\, b \ni \chi_2 = \phi_2' - \phi_2 : \mathcal{S}^2(\mathfrak{g}^0) \to Z(\mathcal{A})$.

$n = 3$ We have $b\overline{E}_3 + \cancel{D\overline{E}_3} + \cancel{E_2 d_3} + \phi\overline{F}_3 = 0$. Again, $E_1 = E_2 = 0$ and and $F_1 = F_2 = 0$ implie $\overline{E}_3 = E_3$ and $\overline{F}_3 = F_3$. Since there is a split $Z\mathcal{G} = \mathrm{Im}\,\phi_1 \oplus \mathrm{Im}\, b$, $bE_3 + \phi_1 F_3 = 0 \Leftrightarrow bE_3 = \phi_1 F_3 = 0$. The map ϕ_1 is injective, so $F_3 = 0$. Using the Lemma 2.5.5, the term E_3 writes $E_3 = \cancel{b\phi_3} + D_2\overline{\phi} - \phi_1 d_3 - \phi_2\overline{d_2}$. Define $\alpha : \mathcal{S}^3(\mathfrak{g}^0) \to \mathfrak{g}^{-1}$, $|\alpha| = 1$, as $\alpha = D_2\overline{\phi} - \phi_2\overline{d_2}$. As $0 = bE_3 = b\alpha - b\phi_1 d_3 = b\alpha$, α takes its value in $Z(\mathcal{A}) = \phi_1(\mathfrak{g}[1]^{-1}) \subset \mathfrak{A}^0$. So the relation $D_2\overline{\phi} - \phi_2\overline{d_2} =: \phi_1 d_3$ defines the map d_3 such that $E_3 = 0$.

$n \geqslant 4$ The map E is of degree $|E| = 1$. Reasoning with degree in $\mathfrak{g}[1] = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$ as in the Lemma 2.5.5, we find $E_n = 0$, and also $F_n = 0$ for $n \geqslant 4$.

In particular, we have $0 = d'\overline{d'}_4$, so

$$0 = (d_2 + d_3)(\overline{d_2} + \overline{d_3}) = \underbrace{d_2\overline{d_2}|_4}_{=0} + d_2\overline{d_3}|_4 + d_3\overline{d_2}|_4 + \underbrace{d_3\overline{d_3}|_4}_{=0},$$

hence $\qquad d_2\overline{d_3}|_4 + d_3\overline{d_2}|_4 = 0.$

Thus we have $0 = d_2\overline{d}_3 + d_3\overline{d}_2 = d_2 \circ_{NR} d_3 + d_3 \circ_{NR} d_2 = [d_2, d_3]_{NR}$.

Computing on arguments $y_i \in \mathfrak{g}[1]^{-1}$, we have

$$\begin{aligned} d_2\overline{d_3}(y_0 \bullet y_1 \bullet y_2 \bullet y_3) =& d_2(d_3 \tilde{\star} id)(y_0 \bullet y_1 \bullet y_2 \bullet y_3) \\ =& d_2\mu_{sh}(d_3 \otimes id)(\mathrm{pr}_3 \otimes \mathrm{pr})\Delta_{sh}(y_0 \bullet y_1 \bullet y_2 \bullet y_3) \\ =& d_2(d_3(y_0 \bullet y_1 \bullet y_2) \bullet y_3) - d_2(d_3(y_0 \bullet y_1 \bullet y_3) \bullet y_2) \\ &+ d_2(d_3(y_0 \bullet y_2 \bullet y_3) \bullet y_1) - d_2(d_3(y_1 \bullet y_2 \bullet y_3) \bullet y_0), \end{aligned}$$

and

$$\begin{aligned} d_3\overline{d_2}(y_0 \bullet y_1 \bullet y_2 \bullet y_3) =& d_3(d_2 \tilde{\star} id)(y_0 \bullet y_1 \bullet y_2 \bullet y_3) \\ =& d_3\mu_{sh}(d_2 \otimes id)(\mathrm{pr}_2 \otimes \mathrm{pr})\Delta_{sh}(y_0 \bullet y_1 \bullet y_2 \bullet y_3) \\ =& d_3(d_2(y_0 \bullet y_1) \bullet y_2 \bullet y_3) - d_3(d_2(y_0 \bullet y_2) \bullet y_1 \bullet y_3) \\ &+ d_3(d_2(y_0 \bullet y_3) \bullet y_1 \bullet y_2) + d_3(d_2(y_1 \bullet y_2) \bullet y_0 \bullet y_3) \\ &- d_3(d_2(y_1 \bullet y_3) \bullet y_0 \bullet y_2) + d_3(d_2(y_2 \bullet y_3) \bullet y_0 \bullet y_1). \end{aligned}$$

Noting $\sigma := \mathfrak{g}^{0^{\wedge 3}} \to \mathfrak{g}^0$ such that $d_3 = \sigma[1]$ and considering the adjoint representation $\rho(x) = ad_x = [x, \ ]_s$, we have

$$\begin{aligned} \delta^3_{CE}\sigma(x_0, x_1, x_2, x_3) =& \sum_{i=0}^{3} \rho(x_i)(\sigma(x_0, \ldots, \widehat{x_i}, \ldots, x_3)) \\ &+ \sum_{0 \leqslant i < j \leqslant 3} (-1)^{i+j}\sigma([x_i, x_j]_s, x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_3) \\ =& -[d_2, d_3]_{NR}(x_0, x_1, x_2, x_3) = 0 \end{aligned}$$

This shows that $d_3[-1] = \sigma$ is a 3-cocycle from $\mathfrak{g}^0$ with values in $Z(\mathcal{A}) = \mathfrak{g}^{-1}$. □

2.5.2 General result

The construction of a perturbed differential to obtain a new contraction between L_∞-algebras is done by induction in [Hue11], Bordemann gives in [Bor] a closed formula.

Theorem 2.5.7 *Let (V, b) be a chain complex and H its cohomology. Suppose that there is a contraction*

$$H \underset{\pi_1}{\overset{\phi_1}{\rightleftarrows}} (V, b) \qquad \circlearrowright h$$

and that D is a perturbation of b, or equivalently that $(V, \overline{b + D})$ is a L_∞-algebra. Then there is a contraction between graded cocommutative symmetric coalgebras

$$(\mathcal{S}(H[1]), \overline{d}) \underset{\overline{\psi}}{\overset{\overline{\varphi}}{\rightleftarrows}} (\mathcal{S}(V[1]), \overline{b + D}) \qquad \circlearrowright \eta$$

the homotopy η *satisfying* $\eta \circ \overline{b+D} \circ \eta = \eta$. *Moreover, setting* $\varphi_1 = \phi_1[1]$ *and* $\psi_1 = \pi_1[1]$, *the maps* $\overline{d}$, $\overline{\varphi}$ *and* $\overline{\psi}$ *are graded coalgebra morphisms and are determined by the formulas*

$$\overline{d} = \overline{\psi_1} \circ (id_{\mathcal{S}(V[1])} + \overline{D} \circ \eta)^{-1} \circ \overline{D} \circ \overline{\varphi_1}, \tag{2.5.5}$$

$$\overline{\varphi} = (id_{\mathcal{S}(V[1])} + \eta \circ \overline{D})^{-1} \circ \overline{\varphi_1}, \tag{2.5.6}$$

$$\overline{\psi} = \overline{\psi_1} \circ (id_{\mathcal{S}(V[1])} + \overline{D} \circ \eta)^{-1}. \tag{2.5.7}$$

This contains in particular that $\overline{\varphi}$ is a L_∞-morphism satisfying

$$\overline{b+D} \circ \overline{\varphi} = \overline{\varphi} \circ \overline{d}.$$

The map η is defined as follow. We set $W := \operatorname{Im} b \oplus \operatorname{Im} h$, so $V = H \oplus W$ and $\mathcal{S}(V[1]) \cong \mathcal{S}(H[1]) \otimes \mathcal{S}(W[1])$. Let $y_1 \bullet \cdots \bullet y_k \in \mathcal{S}(H[1])$ and $w_1 \bullet \cdots \bullet w_l \in \mathcal{S}(W[1])$. Then

$$\eta : \mathcal{S}(H[1]) \otimes \mathcal{S}(W[1]) \to \mathcal{S}(V[1])$$

$$\eta(y_1 \bullet \cdots \bullet y_k \bullet w_1 \bullet \cdots \bullet w_l) = \begin{cases} \overline{h}(y_1 \bullet \cdots \bullet y_k \bullet \frac{1}{l} w_1 \bullet \cdots \bullet w_l) & \text{si } l \neq 0, \\ 0 & \text{si } l = 0, \end{cases} \tag{2.5.8}$$

where $\overline{h}$ is the coderivation induced by h.

Corollary 2.5.8 *Under the hypothesis of the previous theorem, suppose there is another contraction of chain complexes*

$$H' \underset{\pi_1'}{\overset{\phi_1'}{\rightleftarrows}} (V, b) \qquad \circlearrowleft h' .$$

Then, the two L_∞ *structures* $\overline{d}$ *and* $\overline{d'}$ *on* H *and* H' *are conjugated i.e. there is an isomorphism of coalgebras* $\overline{S} : \mathcal{S}(H'[1]) \to \mathcal{S}(H[1])$ *such that*

$$\overline{d'} = \overline{S}^{-1} \circ \overline{d} \circ \overline{S}. \tag{2.5.9}$$

Proof. Using restrictions, since $\overline{\psi'}_1 = \psi_1' = \pi_1'[1]$ induces an isomorphism $H'[1] \to H'[1]$ and $\overline{\varphi}_1 = \varphi_1$ induces an isomorphism $H[1] \to H[1]$, we have that $(\overline{\psi'} \circ \overline{\varphi})|_1 = \psi_1' \varphi_1$ is an isomorphism, which implies that $\overline{\psi'} \circ \overline{\varphi}$ is invertible using Lemma 2.5.9 below.

We have

$$\overline{\psi'} \circ \overline{\varphi} \circ \overline{d} = \overline{\psi'} \circ \overline{b+D} \circ \overline{\varphi} = \overline{d'} \circ \overline{\psi'} \circ \overline{\varphi}$$

hence

$$\overline{d'} = \overline{\psi'} \circ \overline{\varphi} \circ \overline{d} \circ (\overline{\psi'} \circ \overline{\varphi})^{-1}$$

and $\overline{S} := \overline{\psi'} \circ \overline{\varphi}$ fits. □

It remains to prove the lemma about invertibility.

Lemma 2.5.9 *Let $\overline{\varphi} : \mathcal{S}(U) \to \mathcal{S}(V)$ be a graded coalgebra morphism such that $\varphi_1 : U \to V$ is an isomorphism. Then $\overline{\varphi}$ is invertible.*

Proof. Set $\varphi = \varphi_1 + \varphi'$ with $\varphi' = \sum_{k=2}^{\infty} \varphi_k$ where $\varphi_k : \mathcal{S}^k(U) \to V$. We can write

$$\begin{aligned}
\overline{\varphi} &= e^{\tilde{\star}\varphi} = e^{\tilde{\star}(\varphi_1+\varphi')} \\
&= e^{\tilde{\star}\varphi_1} \tilde{\star} e^{\tilde{\star}\varphi'} \\
&= \underbrace{\overline{\varphi_1} \tilde{\star} \left(e^{\tilde{\star}\varphi'} - \mathbf{1}_{\mathcal{S}V}\, \varepsilon_{\mathcal{S}U} \right)}_{\text{loc. nil.}} + \overline{\varphi_1},
\end{aligned}$$

with the first term locally nilpotent. We obtain

$$\overline{\varphi} = \overline{\varphi_1} \circ \left(id_{\mathcal{S}U} + \overline{\varphi_1^{-1}} \left(\overline{\varphi_1} \tilde{\star} \left(e^{\tilde{\star}\varphi'} - \mathbf{1}_{\mathcal{S}V}\, \varepsilon_{\mathcal{S}U} \right) \right) \right)$$

since

$$\begin{aligned}
\mathrm{pr}_U\, \overline{\varphi_1^{-1}} \circ \overline{\varphi_1} &= \varphi_1^{-1}\, \mathrm{pr}_V\, \overline{\varphi_1} = \varphi_1^{-1} \varphi_1\, \mathrm{pr}_U = \mathrm{pr}_U \\
\mathrm{pr}_V\, \overline{\varphi_1} \circ \overline{\varphi_1^{-1}} &= \varphi_1\, \mathrm{pr}_U\, \overline{\varphi_1^{-1}} = \varphi_1 \varphi_1^{-1}\, \mathrm{pr}_V = \mathrm{pr}_V,
\end{aligned}$$

and so

$$\overline{\varphi} = \overline{\varphi_1} \circ \left(id_{\mathcal{S}U} + id_{\mathcal{S}U} \tilde{\star} \left(e^{\tilde{\star}\varphi_1^{-1}\varphi'} - \mathbf{1}_{\mathcal{S}U}\, \varepsilon_{\mathcal{S}U} \right) \right).$$

Thus, the following map is well-defined,

$$\begin{aligned}
\overline{\varphi}^{-1} &= \left(id_{\mathcal{S}U} + id_{\mathcal{S}U} \tilde{\star} \left(e^{\tilde{\star}\varphi_1^{-1}\varphi'} - \mathbf{1}_{\mathcal{S}U}\, \varepsilon_{\mathcal{S}U} \right) \right)^{-1} \circ \overline{\varphi_1^{-1}} \\
&= \sum_{k=0}^{\infty} (-1)^k \left(id_{\mathcal{S}U} \tilde{\star} \left(e^{\tilde{\star}\varphi_1^{-1}\varphi'} - \mathbf{1}_{\mathcal{S}U}\, \varepsilon_{\mathcal{S}U} \right) \right)^k \circ \overline{\varphi_1^{-1}},
\end{aligned}$$

it is the inverse of $\overline{\varphi}$. □

3

Study of the formality for the free algebras

Contents

We study the formality for free algebras on a vector space. We first recall its cohomology, and show that there is formality for spaces of dimension 0 or 1. In greater dimension, we expose the formality equations and finally prove that they can not be satisfied. In finite dimension, we describe the cohomology as the kernel of a trace-form map. Computation of a L_∞ structure gives a 3-arity component to the differential induced by the Schouten bracket (of arity 2) and induces a L_∞-morphism which only has its first two component non zero.

3.1 Description of the spaces

Let $\mathbb{K}$ be a field and V be a $\mathbb{K}$-vector space. We consider the free Lie algebra $\mathcal{L}(V)$ generated by V. Its universal enveloping algebra is isomorphic to the free algebra on V (see [Jac62]),

$$\mathcal{U}(\mathcal{L}(V)) \cong \mathcal{T}V.$$

We will work directly with the free algebra $\mathcal{T}V$.

3.1.1 Definitions

We consider the free algebra $\mathcal{A} := \mathcal{T}V$ generated by V, defined as usual by

$$\mathcal{T}V := \bigoplus_{k=0}^{\infty} V^{\otimes k} =: \bigoplus_{k=0}^{\infty} \mathcal{T}V^k \tag{3.1.1}$$

where $V^{\otimes 0} = \mathbb{K}$, and $V^{\otimes k} = V \otimes \cdots \otimes V$ (k times).

The Hochschild cohomology of $\mathcal{T}V$ with values in a bimodule $\mathcal{M}$ can be computed using a free resolution of $\mathcal{T}V$, it was already known in [CE56, Chap. IX p. 181], for a detailed computation see for example [Hof07, Chap. 5, Prop. 5.3.2].

Theorem 3.1.1 *For $k \in \mathbb{N}$, the k^{th} Hochschild cohomology group of $\mathcal{T}V$ is given by*

$$H_H^k(\mathcal{T}V, \mathcal{M}) = \begin{cases} \mathcal{M}^{\mathcal{T}V} := \{m \in \mathcal{M}, \forall\ a \in \mathcal{T}V, am = ma\} & \text{if } k = 0, \\ H_H^1(\mathcal{T}V, \mathcal{M}) & \text{if } k = 1, \\ \{0\} & \text{if } k \geqslant 2. \end{cases} \tag{3.1.2}$$

Let $\mathcal{M} = \mathcal{T}V$ be the defining bimodule. Then $\mathcal{T}V^{\mathcal{T}V}$ is the centre of $\mathcal{T}V$,

$$\mathcal{T}V^{\mathcal{T}V} = \begin{cases} \mathcal{T}V & \text{if } \dim V = 0 \text{ or } \dim V = 1, \\ \mathbb{K}\mathbf{1} = \mathcal{T}V^0 & \text{if } \dim V \geqslant 2. \end{cases}$$

Note that for $\dim V = 0$, $\mathcal{T}V \cong \mathbb{K}$ and for $\dim V = 1$, $V = \mathbb{K}x$ and $\mathcal{T}V \cong \mathbb{K}[x]$.

Proposition 3.1.2 *For the Hochschild cohomology of $\mathcal{T}V$, the 1-cocycles and 1-coboundaries are of the following form.*

$$\begin{aligned} Z\mathfrak{A}^1(\mathcal{T}V, \mathcal{T}V) &= \mathrm{Der}(\mathcal{T}V, \mathcal{T}V) \\ &:= \{\varphi \in \mathrm{Hom}(\mathcal{T}V, \mathcal{T}V), \varphi(ab) = \varphi(a)b + a\varphi(b)\} \end{aligned} \tag{3.1.3}$$

and

$$B\mathfrak{A}^1(\mathcal{T}V,\mathcal{T}V) = \mathrm{Inder}(\mathcal{T}V,\mathcal{T}V)$$
$$:= \{\varphi \in \mathrm{Hom}(\mathcal{T}V,\mathcal{T}V), \exists\, c \in \mathcal{T}V, \forall\, a \in \mathcal{T}V,\ \varphi(a) = ca - ac\}. \quad (3.1.4)$$

Note that for $\dim V \leqslant 1$, $B\mathfrak{A}^1(\mathcal{T}V,\mathcal{T}V) = \{0\}$.

3.1.2 Examples for spaces of dimension 0 and 1

Let $\mathcal{A} = T\{0\} = \mathbb{K}$.

Proposition 3.1.3 *The Hochschild complex associated to the algebra $\mathcal{A}$ is formal with $\phi_1 = id_{\mathbb{K}} : \mathbb{K} \to \mathfrak{A}^0 \cong \mathbb{K}$ and $\phi_k = 0$ for $k \geqslant 2$.*

Proof. We have $\mathfrak{A}^k = \mathrm{Hom}(\mathbb{K}^{\otimes k},\mathbb{K}) \cong \mathbb{K}$ for all k and $[\ ,\ ]_G = 0$,

$$b : \mathfrak{A}^k \to \mathfrak{A}^{k+1} = \begin{cases} id_{\mathbb{K}} & \text{if } k = 2r+1,\ r \in \mathbb{N}, \\ 0 & \text{if } k = 2r,\ r \in \mathbb{N} \end{cases}$$

and therefore

$$\mathfrak{a}^k = \begin{cases} \mathbb{K} & \text{if } k = 0, \\ 0 & \text{if } k \geqslant 1. \end{cases}$$

□

Let $\mathcal{A} = T(\mathbb{K}x) \cong \mathbb{K}[x]$ be the ring of polynomials in one variable.

Proposition 3.1.4 *The Hochschild complex associated to the algebra $\mathcal{A} = \mathbb{K}[x]$ is formal.*

Proof. We have

$$\mathfrak{a}^0 = \mathcal{A}$$

and

$$\begin{aligned}\mathfrak{a}^1 &= \mathrm{Der}\,\mathcal{A}/\mathrm{Inder}\,\mathcal{A} = \mathrm{Der}\,\mathcal{A} \\ &= \mathrm{Hom}(\mathbb{K},\mathcal{A}) \\ &= \{f\partial_x, f \in \mathcal{A}\}\end{aligned}$$

is the Lie algebra of vector fields.

Let $\phi_1|_{\mathfrak{a}^0} = id_{\mathcal{A}}$ and $\phi_1|_{\mathfrak{a}^1} : \mathfrak{a}^1 \to \mathrm{Hom}(\mathcal{A},\mathcal{A})$ the function which associates to $f\partial_x$ its derivation. Then we have

$$\phi_1([x,y]_s) = [\phi_1(x),\phi_1(y)]_G$$

and with $\phi_k = 0$ for $k \geqslant 2$, the Hochschild complex associated to the algebra $\mathcal{A} = \mathbb{K}[x]$ is formal. □

3.1.3 Results for spaces of dimension greater than 2

We suppose that $\dim V \geqslant 2$.

We have already computed

$$H_H^0(\mathcal{T}V,\mathcal{T}V) = \mathcal{T}V^{\mathcal{T}V} = \mathfrak{a}^0 \cong \mathbb{K}\mathbf{1}.$$

We will consider another description of

$$\mathrm{H_H}^1(\mathcal{T}V,\mathcal{T}V) = \mathrm{Der}(\mathcal{T}V,\mathcal{T}V)/\mathrm{Inder}(\mathcal{T}V,\mathcal{T}V).$$

Cocycles are derivations of $\mathcal{T}V$, which can be seen as maps from V to $\mathcal{T}V$,

$$\mathrm{Der}(\mathcal{T}V,\mathcal{T}V) \cong \mathrm{Hom}(V,\mathcal{T}V). \tag{3.1.5}$$

Indeed, any derivation of $\mathcal{T}V$ is uniquely determined by its values on the generating subspace $\mathcal{T}V^1 = V$. On the other hand, each $\psi \in \mathrm{Hom}(V,\mathcal{T}V)$ uniquely determines a derivation $\overline{\psi}$ of $\mathcal{T}V$ by

$$\begin{aligned} \overline{\psi}(1) &:= 0 \\ \overline{\psi}(v_1\cdots v_k) &:= \sum_{r=1}^{k} v_1\cdots v_{r-1}\psi(v_r)v_{r+1}\cdots v_k. \end{aligned} \tag{3.1.6}$$

Restricted to $\mathcal{T}V$, the differential b is exactly the adjoint operator,

$$\begin{aligned} b : \mathcal{T}V &\to \mathrm{Hom}(\mathcal{T}V,\mathcal{T}V) \\ x &\mapsto ad_x := [x,\cdot] \end{aligned}$$

where $[\cdot,\cdot]$ is the usual commutator. We consider the map $\tilde{b} : \mathcal{T}V \to \mathrm{Hom}(V,\mathcal{T}V)$, where the adjoint takes values only in V. This allows to see the coboundaries $B\mathfrak{A}^1 = \mathrm{Inder}(\mathcal{T}V,\mathcal{T}V) \cong \tilde{b}\mathcal{T}V^+$ as a subspace of $\mathrm{Hom}(V,\mathcal{T}V)$: we take the adjoints derivations coming from elements of $\mathcal{T}V^+$, seen as maps from $\mathrm{Hom}(V,\mathcal{T}V)$. Notice that $\overline{\tilde{b}} = b$.

We have obtained the following result.

Theorem 3.1.5 *For* $\dim V \geqslant 2$, *the Hochschild cohomology of* $\mathcal{T}V$ *is*

$$\begin{aligned} \mathfrak{a} &= \mathfrak{a}^0 \oplus \mathfrak{a}^1 \\ &= \mathcal{T}V^{\mathcal{T}V} \oplus \mathrm{Der}(\mathcal{T}V,\mathcal{T}V)/\mathrm{Inder}(\mathcal{T}V,\mathcal{T}V) \\ &= \mathbb{K}\mathbf{1} \oplus \mathrm{Hom}(V,\mathcal{T}V)/\tilde{b}\mathcal{T}V^+. \end{aligned}$$

The cohomology of $\mathcal{T}V$ concentrates in two levels, so the Theorem 2.5.6 ensures that there is a L_∞-structure on $\mathfrak{g} = \mathfrak{a}[1]$ of the form $d_2 + d_3$, with d_3 a 3-cocycle of the Chevalley-Eilenberg cohomology of $\mathfrak{g}$. In the following, we will show that the Hochschild complex of $\mathcal{T}V$ is not formal for the natural L_∞-structure, and explicit the term d_3.

We will work with the space $Z\mathfrak{A}^1 = \mathrm{Hom}(V,\mathcal{T}V)$ rather than $\mathrm{Der}(\mathcal{T}V,\mathcal{T}V)$, it is a sub-Lie algebra of $\mathrm{Hom}(\mathcal{T}V,\mathcal{T}V)$ with the bracket $[\ ,\ ]_D$: for $\psi,\chi \in \mathrm{Hom}(V,\mathcal{T}V)$, $[\psi,\chi]_D := \overline{\psi}\chi - \overline{\chi}\psi$, where $\overline{\psi}$ and $\overline{\chi}$ are derivations of $\mathcal{T}V$ like in (3.1.6).

$\mathrm{Hom}(V,\mathcal{T}V)$ carries an additional $\mathbb{Z}$-grading according to the degree

$$\mathrm{Hom}(V,\mathcal{T}V)^k = \mathrm{Hom}(V,V^{\otimes k+1}),$$

this grading is auxiliary, no signs are attached. Since we consider the shifted cohomology

$$\begin{aligned}\mathfrak{a}[1] &= \mathfrak{a}[1]^{-1} \oplus \mathfrak{a}[1]^0\\ &= \mathbb{K}1 \oplus \mathrm{Hom}(V,\mathcal{T}V)/\mathcal{T}V^+,\end{aligned}$$

the elements of $\mathrm{Hom}(V,\mathcal{T}V)/\mathcal{T}V^+$ are of degree 0 for the first grading ; thus without ambiguity, we can denote by $|\ |$ the auxiliary grading of $\mathrm{Hom}(V,\mathcal{T}V)$.

$B\mathfrak{A}^1 \cong \mathcal{T}V^+$ also carries the degree of $\mathcal{T}V^+$, ${B\mathfrak{A}^1}^k = V^{\otimes k}$ for $k \geqslant 1$. We have

$$\mathrm{Hom}(V,\mathcal{T}V)^{-1} = \mathrm{Hom}(V,\mathbb{K}) = V^\star, \tag{3.1.7}$$
$${B\mathfrak{A}^1}^{-1} = \{0\},$$
$$\mathrm{Hom}(V,\mathcal{T}V)^0 = \mathrm{Hom}(V,V), \tag{3.1.8}$$
$${B\mathfrak{A}^1}^0 = \{0\}.$$

Here are some example of the Lie bracket of $\mathrm{Hom}(V,\mathcal{T}V)$ which will be used later.

Example 3.1.6 Let $\alpha,\beta \in \mathrm{Hom}(V,\mathcal{T}V)^{-1} = V^\star$, $A,B \in \mathrm{Hom}(V,\mathcal{T}V)^0 = \mathrm{Hom}(V,V)$, $\psi \in \mathrm{Hom}(V,\mathcal{T}V)$, we have

$$\begin{aligned}[\alpha,\beta]_D = 0 \quad &\text{because it is of degree } -2 \ ([\ ,\ ]_D \text{ is of degree } 0),\\ [\alpha,\psi]_D &= \overline{\alpha}\psi,\\ [A,\psi]_D &= \overline{A}\psi - \psi A,\\ [A,B]_D &= AB - BA = [A,B].\end{aligned} \tag{3.1.9}$$

Moreover, since $\tilde{b} = ad$, for $v, x, y \in \mathcal{T}V$,

$$\begin{aligned} \tilde{b}v &= ad_v, \\ [\psi, ad_v]_D &= ad_{\overline{\psi}(v)} = \tilde{b}\overline{\psi}(v), \\ [ad_x, ad_y]_D &= ad_{[x,y]} = \tilde{b}[x,y]. \end{aligned} \tag{3.1.10}$$

We set

$$\begin{aligned} \mathcal{H}^{-1} &= V^\star, \\ \mathcal{H}^0 &= \mathrm{Hom}(V,V), \end{aligned}$$

and choose in each $\mathrm{Hom}(V, V^{\otimes k+1})$ a complementary subspace to the inner derivations, *i.e.* $\mathrm{Hom}(V, V^{\otimes k+1}) = \mathcal{H}^k \oplus B{\mathfrak{A}^1}^k$. We denote by $\mathcal{H} = \bigoplus_{n \geqslant -1} \mathcal{H}^n$ the graded complement of $B\mathfrak{A}^1 = b\mathcal{T}V$. Let

$$P_k : \mathrm{Hom}(V, V^{\otimes k+1}) \to \mathcal{H}^k \tag{3.1.11}$$

be the canonical projection, and for $k \geqslant 1$, let

$$Q_k : \mathrm{Hom}(V, V^{\otimes k+1}) \to V^{\otimes k} \tag{3.1.12}$$

be the canonical map such that $id - P_k = bQ_k$. Set $Q_{-1} = Q_0 = 0$. Let $P = \sum_{k \geqslant -1} P_k$, $Q = \sum_{k \geqslant 1} Q_k$. Note that for $x, y \in \mathcal{H}$, $P([x,y]_D) = [\overline{x}, \overline{y}]_s$, where $\overline{x}$ and $\overline{y}$ are the canonical projection of x and y in the cohomology $\mathrm{Hom}(V, \mathcal{T}V)/\mathcal{T}V^+$.

In fact, endowed with the bracket $[\ ,\ ]_D$, $\mathfrak{A}_{\mathrm{red}} := \mathcal{T}V \oplus \mathrm{Hom}(V, \mathcal{T}V)$ is a graded Lie subalgebra of the Hochschild complex $\mathfrak{A} := \mathrm{C_H}(\mathcal{T}V, \mathcal{T}V)$ and there is a contraction

$$(\mathfrak{A}_{\mathrm{red}}, \tilde{b}) \underset{P}{\overset{i}{\rightleftarrows}} (\mathfrak{A}, b) \quad \circlearrowright Q,$$

because $id - P = \tilde{b}Q$. Here we can continue without using the Theorem 2.5.7.

Let $\phi_1 : \mathfrak{a} \to \mathfrak{A}$ be the section according to the previous decomposition. Since $\phi_1(\mathfrak{a}^0 = \mathbb{K}\mathbf{1}) \subset \mathfrak{A}^0$ and $b\phi_1(\mathbf{1}) = 0$, it follows that, for all $a \in \mathcal{T}V$, $a\phi_1(\mathbf{1}) - \phi_1(\mathbf{1})a = 0$ so $\phi_1(\mathbf{1}) = \mathbf{1}$ because $p(\mathbf{1}) = \mathbf{1}$.

To study the formality of the Hochschild complex of $\mathcal{T}V$, we must verify the equation (2.2.8) for $0 \leqslant k \leqslant 2$. On level $k = 0$, we see that the equation (2.2.4) is satisfied by construction. On level $k = 1$, we have the equation (2.2.5)

$$\forall\ \xi_1, \xi_2 \in \mathfrak{a}[1],\ \phi_1([\xi_1, \xi_2]_s) = b\phi_2(\xi_1, \xi_2) + [\phi_1(\xi_1), \phi_1(\xi_2)]_G$$

which vanishes on both sides if one of the ξ_i is in $\mathfrak{a}^0 = \mathbb{K}\mathbf{1}$. To lighten computations, we will work in $\mathcal{H}$ instead of $\mathrm{Hom}(V, \mathcal{T}V)/\mathcal{T}V^+$. For $\xi_1 = \overline{x}, \xi_2 = \overline{y} \in \mathfrak{a}^1$, $x, y \in \mathcal{H}$, the equation in $\mathrm{Hom}(V, \mathcal{T}V)$ becomes

$$P([x,y]_D) = b\phi_2(x,y) + [x,y]_D \tag{3.1.13}$$

hence

$$\phi_2(x,y) = -Q([x,y]_D) + q(x,y)\mathbf{1} \tag{3.1.14}$$

where $q : \mathcal{H} \wedge \mathcal{H} \to \mathbb{K}$ is arbitrary, since $bq(x,y)\mathbf{1} = 0$.

Let us look at level $k = 2$. Let $x_1, x_2, x_3 \in \mathcal{H}$. If the Hochschild complex of $\mathcal{T}V$ is formal, since $\phi_3 = 0$ the equation (2.2.8) would become

$$\circlearrowleft_{x_1,x_2,x_3} \phi_2(P[x_1,x_2]_D, x_3) - [x_1, \phi_2(x_2,x_3)]_D = 0$$

However, using the equation (3.1.14), we obtain

$$\begin{aligned}
&\circlearrowleft_{x_1,x_2,x_3} \left[\phi_2(P[x_1,x_2]_D, x_3) - \overline{x_1}(\phi_2(x_2,x_3))\right] \\
&= \circlearrowleft_{x_1,x_2,x_3} \left[-Q([P[x_1,x_2]_D,x_3]_D) + q(P[x_1,x_2]_D,x_3) + \overline{x_1}(Q[x_2,x_3]_D)\right] \\
&=: T(x_1,x_2,x_3) \in \mathcal{T}V
\end{aligned} \tag{3.1.15}$$

This element is a scalar, to see this, we will compute that $\tilde{b}(T(x_1,x_2,x_3)) = 0$. We have

$$\begin{aligned}
\tilde{b}(T(x_1,x_2,x_3)) &= \circlearrowleft_{x_1,x_2,x_3} \tilde{b}\big(q(P[x_1,x_2]_D,x_3) - Q([P[x_1,x_2]_D,x_3]_D)\big) + \tilde{b}(\overline{x_1}(Q[x_2,x_3]_D)) \\
&= \circlearrowleft_{x_1,x_2,x_3} \left[(P - id)([P[x_1,x_2]_D,x_3]_D) + \tilde{b}(\overline{x_1}(Q[x_2,x_3]_D))\right]
\end{aligned}$$

because $\operatorname{Im} q \subset \operatorname{Ker} \tilde{b}$ and $\tilde{b}Q = id - P$. Moreover, since for $v \in V$, $w \in \mathcal{T}V$, we have $\tilde{b}\overline{x_1}(w)(v) = [\overline{x_1}(w), v] = \overline{x_1}([w,v]) - [w, \overline{x_1}(v)]$, we get

$$\tilde{b}(\overline{x_1}(Q[x_2,x_3]_D))(v) = \overline{x_1}(\tilde{b}(Q[x_2,x_3]_D)(v)) - b(Q[x_2,x_3]_D)(x_1(v)).$$

Thus,

$$\begin{aligned}
\tilde{b}(\overline{x_1}(Q[x_2,x_3]_D)) &= \overline{x_1} \circ \tilde{b}Q[x_2,x_3]_D - \overline{\tilde{b}Q[x_2,x_3]_D} \circ x_1 \\
&= [x_1, \tilde{b}Q[x_2,x_3]_D]_D \\
&= [x_1,[x_2,x_3]_D]_D - [x_1, P[x_2,x_3]_D]_D.
\end{aligned}$$

We then have

$$\begin{aligned}
\tilde{b}T(x_1,x_2,x_3) &= \circlearrowleft_{x_1,x_2,x_3} \Big(P([P[x_1,x_2]_D,x_3]_D) - [P[x_1,x_2]_D,x_3]_D \\
&\qquad + [x_1,[x_2,x_3]_D]_D + [P[x_2,x_3]_D,x_1]_D\Big) \\
&= \circlearrowleft_{x_1,x_2,x_3} \big([[x_1,x_2]_s,x_3]_s + [x_1,[x_2,x_3]_D]_D\big) \\
&= 0
\end{aligned}$$

using Jacobi identity for the Lie brackets $[\ ,\ ]_s$ and $[\ ,\ ]_D$, the other terms cancelling by circular permutation.

Since $\mathcal{T}V = \mathcal{T}V^+ \oplus \operatorname{Ker}\varepsilon$ where ε is the counit of $\mathcal{T}V$, and that $\mathbb{K} \ni T(x_1,x_2,x_3) = \varepsilon(T(x_1,x_2,x_3))$, equation (3.1.15) becomes

$$\begin{aligned}&\circlearrowleft_{x_1,x_2,x_3} \left[\phi_2(P[x_1,x_2]_D,x_3) - \overline{x_1}(\phi_2(x_2,x_3))\right]\\ &= \circlearrowleft_{x_1,x_2,x_3} \left[q([x_1,x_2]_s,x_3) + \varepsilon(\overline{x_1}(Q[x_2,x_3]_D))\right].\end{aligned} \tag{3.1.16}$$

In the equation (3.1.16), the firsts term in q of the right-hand side are a scalar 3-coboundary of the Chevalley-Eilenberg cohomology of $\mathfrak{a}$, up to a minus sign.

As Q is a linear map which is homogeneous of degree 0 (for the grading of $\operatorname{Hom}(V,\mathcal{T}V) = \bigoplus_{k\geqslant -1}\operatorname{Hom}(V,V^{\otimes k+1})$) since $\operatorname{Im}Q \subset \mathcal{T}V^+ = \bigoplus_{k\geqslant 1} V^{\otimes k}$, we have that

$$\varepsilon\big(x_1(Q[x_2,x_3]_D)\big) = 0 \text{ if } |x_1| \geqslant 0 \text{ or if } |[x_2,x_3]_D| \neq 1. \tag{3.1.17}$$

Hence, $\varepsilon\big(\overline{x_1}(Q[x_2,x_3]_D)\big)$ is not 0 a priori if and only if

Cas 1

$$\begin{aligned}&|x_1| = -1 \quad \text{and} \quad |x_2| = 0 \quad \text{and} \quad |x_3| = 1\\ \text{or}\quad &|x_1| = -1 \quad \text{and} \quad |x_2| = 1 \quad \text{and} \quad |x_3| = 0\end{aligned} \tag{3.1.18}$$

Cas 2

$$\begin{aligned}&|x_1| = -1 \quad \text{and} \quad |x_2| = -1 \quad \text{and} \quad |x_3| = 2\\ \text{or}\quad &|x_1| = -1 \quad \text{and} \quad |x_2| = 2 \quad \text{and} \quad |x_3| = -1\end{aligned} \tag{3.1.19}$$

Define

$$\begin{aligned}\sigma\ :\ &\mathfrak{a}\wedge\mathfrak{a}\wedge\mathfrak{a} \to \mathbb{K}\\ &(x_1,x_2,x_3) \mapsto \circlearrowleft_{x_1,x_2,x_3} \varepsilon\Big(\mathsf{pr}_{-1}(x_1)(Q[x_2,x_3]_D)\Big),\end{aligned} \tag{3.1.20}$$

where $\mathsf{pr}_{-1} : \operatorname{Hom}(V,\mathcal{T}V) \to \operatorname{Hom}(V,\mathcal{T}V)^{-1} = V^\star$ is the canonical projection.

In general, in the equation (3.1.16), the coboundary $(\delta_{CE}q)$ does not cancel with the term σ. However we have the following property for σ.

Proposition 3.1.7 *σ is a scalar 3-cocycle of the Chevalley-Eilenberg cohomology of $\mathfrak{a}$.*

Proof. We have to verify that

$$\begin{aligned} 0 &= (\delta_{CE}\sigma)(x_0,x_1,x_2,x_3) \\ &= -\sigma([x_0,x_1]_s,x_2,x_3)+\sigma([x_0,x_2]_s,x_1,x_3)-\sigma([x_0,x_3]_s,x_1,x_2) \\ &\quad -\sigma([x_1,x_2]_s,x_0,x_3)+\sigma([x_1,x_3]_s,x_0,x_2)-\sigma([x_2,x_3]_s,x_0,x_1) \end{aligned}$$

Since σ and $[\ ,\ ]$ are of degree 0, so is $(\delta_{CE}\sigma)$, hence we must have $|x_0|+|x_1|+|x_2|+|x_3| = 0$; whitout loss of generality, we can assume $|x_0| \leqslant |x_1| \leqslant |x_2| \leqslant |x_3|$. There are five cases :

Case 1 $|x_i| \geqslant 0$ for $0 \leqslant i \leqslant 3$, hence $|x_i| = 0$ for $0 \leqslant i \leqslant 3$.
We get $(\delta\sigma)(x_0,x_1,x_2,x_3) = 0$ since $\sigma(y_1,y_2,y_3) = 0$ if $|y_i| = 0,\ 1 \leqslant i \leqslant 3$.

Case 2 $|x_0| = -1,\ |x_i| \geqslant 0$ for $1 \leqslant i \leqslant 3$, so $|x_1| = 0 = |x_2|,\ |x_3| = 1$.

Case 3 $|x_0| = -1 = |x_1|,\ |x_2| = 0,\ |x_3| = 2$.

Case 4 $|x_0| = -1 = |x_1|,\ |x_2| = 1 = |x_3|$.

Case 5 $|x_0| = |x_1| = |x_2| = -1,\ |x_3| = 3$.

We show that $(\delta_{CE}\sigma)(x_0,x_1,x_2,x_3)$ vanishes in the third case, the other are similar (except the first, which is direct).

We note $x_0 = \alpha,\ x_1 = \beta,\ x_2 = A$ and $x_3 = \varphi$.

$$\begin{aligned} (\delta_{CE}\sigma)(\alpha,\beta,A,\varphi) = &-\sigma([\alpha,\beta]_D,A,\varphi)+\sigma([\alpha,A]_D,\beta,\varphi)-\sigma([\alpha,\varphi]_s,\beta,A) \\ &-\sigma([\beta,A]_D,\alpha,\varphi)+\sigma([\beta,\varphi]_s,\alpha,A)-\sigma([A,\varphi]_s,\alpha,\beta) \end{aligned}$$

Using the equations (3.1.9), (3.1.10), (3.1.17) and then the decomposition $[x,y]_s = P[x,y]_D = [x,y]_D - bQ[x,y]_D$, we get

$$\begin{aligned} (\delta_{CE}\sigma)(\alpha,\beta,A,\varphi) &= \varepsilon\big(\alpha A(Q[\beta,\varphi]_D)\big)+\varepsilon\big(\beta(Q[\varphi,\alpha A]_D)\big)-\varepsilon\big(\beta(Q[A,[\alpha,\varphi]_s]_D)\big) \\ &\quad -\varepsilon\big(\beta A(Q[\alpha,\varphi]_D)\big)-\varepsilon\big(\alpha(Q[\varphi,\beta A]_D)\big)+\varepsilon\big(\alpha(Q[A,[\beta,\varphi]_s]_D)\big) \\ &\quad -\varepsilon\big(\alpha(Q[\beta,[A,\varphi]_s]_D)\big)-\varepsilon\big(\beta(Q[[A,\varphi]_s,\alpha]_D)\big) \\ &= \varepsilon\big(\alpha A(Q[\beta,\varphi]_D)\big)+\varepsilon\big(\beta(Q([\varphi,[\alpha,A]_D]_D+[A,[\varphi,\alpha]_D]_D+[\alpha,[A,\varphi]_D]_D))\big) \\ &\quad -\varepsilon\big(\alpha(Q([\varphi,[\beta,A]_D]_D+[A,[\varphi,\beta]_D]_D+[\beta,[A,\varphi]_D]_D))\big) \\ &\quad +\varepsilon\big(\beta(Q[A,\tilde{b}Q[\alpha,\varphi]_D]_D)\big)-\varepsilon\big(\beta(Q[\alpha,\tilde{b}Q[A,\varphi]_D]_D)\big) \\ &\quad -\varepsilon\big(\alpha(Q[A,\tilde{b}Q[\beta,\varphi]_D]_D)\big)+\varepsilon\big(\alpha(Q[\beta,\tilde{b}Q[A,\varphi]_D]_D)\big)-\varepsilon\big(\beta A(Q[\alpha,\varphi]_D)\big). \end{aligned}$$

The second and third terms vanish by the Jacobi relation. Moreover, for $x,y \in \mathrm{Hom}(V,\mathcal{T}V)$, $\tilde{b}[x,Q(y)]_D = [x,\tilde{b}Q(y)] = [x,(id{-}P)(y)]_D$

which implies that $[x,Q(y)]_D = Q[x,(id-P)(y)]_D + \varepsilon([x,Q(y)]_D)$. Adding the fact that $\alpha(\mathbf{1}) = 0 = \beta(\mathbf{1})$, we have

$$\begin{aligned}(\delta_{CE}\sigma)(\alpha,\beta,A,\varphi) &= \varepsilon\big(\alpha A(Q[\beta,\varphi]_D)\big) - \varepsilon\big(\beta A(Q[\alpha,\varphi]_D)\big) \\ &\quad + \varepsilon\big(\beta([A,Q[\alpha,\varphi]_D]_D)\big) - \varepsilon\big(\beta([\alpha,Q[A,\varphi]_D]_D)\big) \\ &\quad - \varepsilon\big(\alpha([A,Q[\beta,\varphi]_D]_D)\big) + \varepsilon\big(\alpha([\beta,Q[A,\varphi]_D]_D)\big) \\ &= 0.\end{aligned}$$

Indeed, the projection by Q gives an element of V, so the action of the bracket on elements of degree 0 is the natural evaluation, for example $[A,Q[\alpha,\varphi]_D]_D = AQ[\alpha,\varphi]_D$, and since $[\alpha,\beta]=0$ the other terms cancel by the Jacobi relation. □

Proposition 3.1.8 *If we choose another graded complement $\mathcal{H}'$ of $B\mathfrak{A}^1$ and therefore another section ϕ_1', the 3-cocycle σ changes by a coboundary.*

Proof. Let $\phi_1' : \mathfrak{a} \to \mathfrak{A}$ be another section associated with the graded complement $\mathcal{H}'$. Taking the canonical projection p of the space of cocycles onto $\mathfrak{a}$, we get $p(\phi_1' - \phi_1) = 0$, so $\phi_1' - \phi_1 \in \operatorname{Im}\tilde{b}$ and there exists $\chi : \mathfrak{a} \to B\mathfrak{A} \cong \mathcal{T}V^+$ such that $\phi_1' = \phi_1 + \tilde{b}\chi$. Denote by P' the projection of $Z\mathfrak{A}^1$ on $\mathcal{H}'$ and Q' the map such that $\tilde{b}Q' = id - P'$. We have $P' = \phi_1' \circ p = (\phi_1 + \tilde{b}\chi)\circ p = P + \tilde{b}\chi p$ so $\tilde{b}Q' = id - P' = id - P - \tilde{b}\chi p = \tilde{b}Q - \tilde{b}\chi p$. Then $\tilde{b}(Q' - (Q - \chi p)) = 0$ and $Q' = Q - \chi p$ since $\tilde{b}$ is one-to-one on $\mathcal{T}V^+$.

Let $\overline{x_1},\overline{x_2},\overline{x_3} \in \mathfrak{a}$, so $\phi_1(\overline{x_1}),\phi_1(\overline{x_2}),\phi_1(\overline{x_3}) \in \mathcal{H}$. As before, pr_{-1} is the projection on $V^\star = \mathcal{H}^{-1} = \mathcal{H}'^{-1}$, so we have for the cocycle σ the following expression :

$$\sigma(\overline{x_1},\overline{x_2},\overline{x_3}) = \underset{x_1,x_2x_3}{\circlearrowleft} \varepsilon\big(\mathrm{pr}_{-1}(\overline{x_1})(Q[\phi_1(\overline{x_2}),\phi_1(\overline{x_3})]_D)\big).$$

We compute the cocycle σ' associated with the graded complement $\mathcal{H}'$.

$$\begin{aligned}\sigma'(\dot{x_1},\dot{x_2},\dot{x_3}) &:= \underset{x_1,x_2x_3}{\circlearrowleft} \varepsilon\big(\mathrm{pr}_{-1}(\dot{x_1})(Q'[\phi_1'(\dot{x_2}),\phi_1'(\dot{x_3})]_D)\big) \\ &= \underset{x_1,x_2x_3}{\circlearrowleft} \varepsilon\big(\mathrm{pr}_{-1}(\dot{x_1})((Q-\chi p)[(\phi_1+\tilde{b}\chi)(\dot{x_2}),(\phi_1+\tilde{b}\chi)(\dot{x_3})]_D)\big) \\ &= \underset{x_1,x_2x_3}{\circlearrowleft} \varepsilon\big(\mathrm{pr}_{-1}(\dot{x_1})((Q-\chi p)([\phi_1(\dot{x_2}),\phi_1(\dot{x_3})]_D + [\phi_1(\dot{x_2}),\tilde{b}\chi(\dot{x_3})]_D \\ &\qquad + [\tilde{b}\chi(\dot{x_2}),\phi_1(\dot{x_3})]_D + [\tilde{b}\chi(\dot{x_2}),\tilde{b}\chi(\dot{x_3})]_D))\big)\end{aligned}$$

By definition, we have $p[\phi_1(\overline{u}),\phi_1(\overline{v})]_D = [\overline{u},\overline{v}]_s$; $p\circ\tilde{b} = 0$ implies $\chi p \tilde{b} = 0$ and with the equations (3.1.9) and (3.1.10), we have

$$\begin{aligned}\sigma'(\dot{x_1},\dot{x_2},\dot{x_3}) =& \varepsilon\big(\mathsf{pr}_{-1}(\dot{x_1})(Q[\phi_1(\dot{x_2}),\phi_1(\dot{x_3})]_D)\big) - \varepsilon\big(\mathsf{pr}_{-1}(\dot{x_1})(\chi[\dot{x_2},\dot{x_3}]_s)\big)\\ &+ \varepsilon\big(\mathsf{pr}_{-1}(\dot{x_1})(Q\tilde{b}(\phi_1(\dot{x_2})(\chi(\dot{x_3}))))\big) - \varepsilon\big(\mathsf{pr}_{-1}(\dot{x_1})(Q\tilde{b}(\phi_1(\dot{x_3})(\chi(\dot{x_2}))))\big)\\ &+ \varepsilon\big(\mathsf{pr}_{-1}(\dot{x_1})(Q\tilde{b}[\chi(\dot{x_2}),\chi(\dot{x_3})]_D)\big) + \text{permutation cyclique des } x_i\end{aligned}$$

and since $Q\tilde{b} = id_{\mathcal{G}|\mathcal{T}V^+}$, and $\|[\mathcal{T}V^+,\mathcal{T}V^+]\| \geqslant 2$,

$$\begin{aligned}\sigma'(\dot{x_1},\dot{x_2},\dot{x_3}) &= \sigma(\dot{x_1},\dot{x_2},\dot{x_3})\\ -\varepsilon\big(\mathsf{pr}_{-1}(\dot{x_1})(\chi[\dot{x_2},\dot{x_3}]_s)\big) &+ \varepsilon\big(\mathsf{pr}_{-1}(\dot{x_1})(\phi_1(\dot{x_2})(\chi(\dot{x_3})))\big)\\ -\varepsilon\big(\mathsf{pr}_{-1}(\dot{x_1})(\phi_1(\dot{x_3})(\chi(\dot{x_2})))\big) &+ \text{permutation cyclique des } x_i\end{aligned}$$

Let $\omega : \mathfrak{a}\wedge\mathfrak{a} \to \mathbb{K}$ defined by $\omega(\overline{x},\overline{y}) = \varepsilon\big(\mathsf{pr}_{-1}(\overline{x})(\chi(\overline{y}))\big) - \varepsilon\big(\mathsf{pr}_{-1}(\overline{y})(\chi(\overline{x}))\big)$. Since $\mathsf{pr}_{-1}([\overline{x},\overline{y}]_s) = \mathsf{pr}_{-1}(\overline{x})\phi_1(\overline{y}) - \mathsf{pr}_{-1}(\overline{y})\phi_1(\overline{x})$, we have

$$\sigma'(\overline{x_1},\overline{x_2},\overline{x_3}) = \sigma(\overline{x_1},\overline{x_2},\overline{x_3}) - (\delta_{CE}\omega)(\overline{x_1},\overline{x_2},\overline{x_3}).$$

□

3.1.4 Case of a finite dimensional space

We suppose that V is of finite dimension $N \geqslant 2$. We can describe $\mathcal{H}$, the graded complement of $\tilde{b}\mathcal{T}V^+$ as a kernel.

Let $\{e_i\}_{1\leqslant i\leqslant N}$ be a base of V and $\{e^i\}_{1\leqslant i\leqslant N}$ its dual basis of $V^\star$. For $n \in \mathbb{N}$ write the applications $\varphi \in \mathrm{Hom}(V, V^{\otimes n+1})$ as

$$\varphi = \sum_{j,i_0,\dots,i_n} \varphi_j^{i_0\dots i_n} e_{i_0}\otimes\cdots\otimes e_{i_n}\otimes e^j.$$

For each $n \in \mathbb{N}$, we consider the map

$$\begin{aligned}S_n : \mathrm{Hom}(V, V^{\otimes n+1}) &\to V^{\otimes n}\\ S_n(\varphi) &\mapsto \sum_{j,i_1,\dots,i_n} \varphi_j^{j i_1\dots i_n} e_{i_1}\otimes\cdots\otimes e_{i_n}.\end{aligned} \tag{3.1.21}$$

Let $S : \mathrm{Hom}(V,\mathcal{T}V) \to \mathcal{T}V$ be the sum $S := \sum_{n\geqslant 0} S_n$, it is homogeneous of degree 0.

Proposition 3.1.9 *Let $n \in \mathbb{N}$.*

(i) $\mathrm{Ker}\, S_n \cap \tilde{b}V^{\otimes n} = \{0\}$.

(ii) For $n \geqslant 1$, $\mathrm{Hom}(V, V^{\otimes n+1}) = \mathrm{Ker}\, S_n \oplus \tilde{b}V^{\otimes n}$, *so* $\mathcal{H}^n \cong \mathrm{Ker}\, S_n$.

Proof. (i) Let $w \in V^{\otimes n}$, we compute, for later use, $S_n(\tilde{b}w) = S_n(ad(w))$. For all $v \in V$, we have

$$\begin{aligned} ad(w)(v) &= wv - vw \\ &= \sum_{i_0,\dots,i_n} \left(w^{i_1\dots i_n} v^{i_0} e_{i_1} \otimes \dots \otimes e_{i_n} \otimes e_{i_0} - v^{i_0} w^{i_1\dots i_n} e_{i_0} \otimes \dots \otimes e_{i_n} \right) \\ &= \sum_{i_0,\dots,i_n} \left(\left(w^{i_0\dots i_{n-1}} \delta_j^{i_n} - \delta_j^{i_0} w^{i_1\dots i_n} \right) v^j e_{i_0} \otimes \dots \otimes e_{i_n} \right), \end{aligned}$$

so $S_n(ad(w)) = \sum_{i_1,\dots,i_n} w^{i_n i_1 \dots i_{n-1}} - N w^{i_1\dots i_n}$, which we can write $S_n(ad(w)) = \zeta(w) - Nw$, were $\zeta(e_{i_0}\dots e_{i_n}) := e_{i_1} \otimes \dots \otimes e_{i_n} \otimes e_{i_0}$ is a cyclic permutation extended as a linear map. Since $\zeta^n = id_{V^{\otimes n}}$, it follows that ζ is diagonalisable over the algebra, with eigenvalues $1, u, \dots, u^{n-1}$ where u is a primitive root of unity. Now $|u| = 1$ and $N \geqslant 2$, hence $\zeta - Nid$ is invertible with

$$\begin{aligned} (\zeta - Nid)^{-1} = \left(-N \left(id - \frac{1}{N}\zeta \right) \right)^{-1} &= -\frac{1}{N} \sum_{i=0}^{\infty} \left(\frac{1}{N}\zeta \right)^i \\ &= -\frac{1}{N} \left(id + \frac{1}{N}\zeta + \dots + \frac{1}{N^{n-1}}\zeta^{n-1} \right) \sum_{i=0}^{\infty} \frac{1}{N^{in}} \\ &= -\frac{1}{N} \frac{1}{1 - \frac{1}{N^n}} \left(id + \frac{1}{N}\zeta + \dots + \frac{1}{N^{n-1}}\zeta^{n-1} \right). \end{aligned}$$

Hence $\operatorname{Ker} S_n \cap \tilde{b} V^{\otimes n} = \{0\}$ and S_n is surjective.

(ii) We have $\dim \operatorname{Hom}(V, V^{\otimes n+1}) = N^{n+2}$, $\dim \tilde{b} V^{\otimes n} = N^n$ for $n \geqslant 1$ and $\dim \operatorname{Ker} S_n = N^{n+2} - N^n$ since S_n is surjective. Hence

$$\begin{aligned} \dim(\operatorname{Ker} S_n + \tilde{b} V^{\otimes n}) &= \dim(\operatorname{Ker} S_n \oplus \tilde{b} V^{\otimes n}) \\ &= N^{n+2} - N^n + N^n = N^{n+2} \\ &= \dim \operatorname{Hom}(V, V^{\otimes n+1}), \end{aligned}$$

and $\operatorname{Hom}(V, V^{\otimes n+1}) = \operatorname{Ker} S_n \oplus \tilde{b} V^{\otimes n} = \mathcal{H}^n \oplus \tilde{b} V^{\otimes n}$, so $\mathcal{H}^n \cong \operatorname{Ker} S_n$.

□

This shows that $\mathcal{H} = \bigoplus_{n \geqslant -1} \operatorname{Ker} S_n$ is a graded complement to $\tilde{b} \mathcal{T} V^+$ and thus serves as a section.

Lemma 3.1.10 *Let $n \in \mathbb{N}$.*

(i) For all $A \in \operatorname{Hom}(V, V)$, $\varphi \in \operatorname{Hom}(V, V^{\otimes n+1})$, $S_n([A, \varphi]) = A(S_n(\varphi))$. In particular, $[A, \operatorname{Ker} S_n] \subset \operatorname{Ker} S_n$ for all $n \in \mathbb{N}$.

(ii) *For all* $\alpha \in V^\star$ *and* $\varphi \in \mathrm{Hom}(V, V^{\otimes n+1})$, $S_{n-1}([\alpha,\varphi]) = \alpha(S_n(\varphi)) + \sum_{j,i_0,i_2\dots,i_n} \alpha_{i_0} \varphi_j^{i_0 j i_2 \dots i_n} e_{i_2} \otimes \cdots \otimes e_{i_n}$.

Proof. (i) We have

$$[A,\varphi]_j^{i_0\dots i_n} = \sum_k \left(A_k^{i_0} \varphi_j^{k i_1 \dots i_n} + \sum_{r=1}^n A_k^{i_r} \varphi_j^{i_0 \dots i_{r-1} k i_{r+1} \dots i_n} - \varphi_k^{i_0 \dots i_n} A_j^k \right),$$

hence

$$S([A,\varphi])^{i_1\dots i_n} = \cancel{\sum_k \left(A_k^j \varphi_j^{k i_1 \dots i_n} \right)} + \sum_k \sum_{r=1}^n A_k^{i_r} \varphi_j^{j i_1 \dots i_{r-1} k i_{r+1} \dots i_n} - \cancel{\sum_k \left(\varphi_k^{j i_1 \dots i_n} A_j^k \right)} = \left(A(S(\varphi)) \right)^{i_1 \dots i_n}$$

(ii) We have $[\alpha,\varphi]_j^{i_0\dots i_{n-1}} = \sum_k \left(\alpha_k \varphi_j^{k i_0 \dots i_{n-1}} + \sum_{r=1}^n \alpha_k \varphi_j^{i_0 \dots i_{r-1} k i_{r+1} \dots i_{n-1}} \right)$ hence

$$\begin{aligned} S_{n-1}([\alpha,\varphi])^{i_1\dots i_{n-1}} &= \sum_{k,j} \left(\alpha_k \varphi_j^{k j i_1 \dots i_{n-1}} + \alpha_k \varphi_j^{j k i_1 \dots i_{n-1}} + \sum_{r=2}^n \alpha_k \varphi_j^{j i_1 \dots i_{r-1} k i_{r+1} \dots i_{n-1}} \right) \\ &= \sum_{k,j} \left(\alpha_k \varphi_j^{k j i_1 \dots i_{n-1}} \right) + \alpha(S_n(\varphi))^{i_1 \dots i_{n-1}} \end{aligned}$$

□

3.2 Perturbed formality

We consider again the equation (3.1.16). If the Hochschild complex associated with $\mathcal{T}V$ was formal, we should find an application q such that the 3-cocycle σ cancels with the 3-coboundary $\delta_{CE} q$. In that case, the formality equation (2.2.8) would be satisfied at the level $k = 2$. However, we will show that this is not the case, meaning that we have the following result.

Theorem 3.2.1 *The cohomology class of the cocycle* σ *does not vanish in* $\mathfrak{g}$.

Corollary 3.2.2 *Let* V *un vectorial space,* $\dim V \geqslant 2$. *The Hochschild complex of* $\mathcal{A} = \mathcal{T}V$ *is not formal.*

The formality equation stay perturbed by the 3-cocycle σ, so we only have a L_∞-morphism between $(\mathfrak{g}, \overline{d'} = \overline{d_2 + \sigma})$ and $(\mathcal{G}, \overline{b + D})$.

The end of this section is the proof of theses results. First, the finite-dimensional case is done mainly by computations; in any dimension (greater than 2) an argument of splitting will be used to obtain the same result in general.

We suppose that V is of finite dimension $N \geqslant 2$. With the equation (3.1.17), we have only the two cases (3.1.18) and (3.1.19) to consider. We obtain more precisely conditions on σ and q in each case.

3.2.1 Computations for the cocycle part in the case (3.1.18)

Lemma 3.2.3 *Let $x_1, x_2, x_3 \in \mathcal{H}$. In the case (3.1.18), $\sigma(x_1, x_2, x_3)$ vanishes.*

Proof. Let $x_1 = \alpha \in \mathcal{H}^{-1} = V^\star$, $x_2 = A \in \mathcal{H}^0 = \mathrm{Hom}(V, V)$ and $x_3 = \varphi \in \mathcal{H}^1$. We want to show that

$$0 = \sigma(\alpha, A, \varphi) = \varepsilon\big(\mathsf{pr}_{-1}(\alpha)(Q[A,\varphi]_D)\big) + \varepsilon\big(\mathsf{pr}_{-1}(A)(Q[\varphi,\alpha]_D)\big) + \varepsilon\big(\mathsf{pr}_{-1}(\varphi)(Q[\alpha,A]_D)\big).$$

Since $[\varphi, \alpha]_D = -\alpha \circ \varphi$ has degree 0 and $Q_0 = 0$, we have that $Q[\varphi, \alpha]_D = 0$ and the second term of the sum vanishes. Similary, since $[\alpha, A]_D = \alpha \circ A$ has degree -1 and $Q_{-1} = 0$, $Q[\alpha, A]_D = 0$ and the third term also vanishes. Now $[A, \varphi]_D$ is of degree 1, but according to Lemma 3.1.10, $[A, \varphi]_D \in \operatorname{Ker} S_1 = \mathcal{H}^1$, so the first term vanishes. □

3.2.2 Computations for the cocycle part in the case (3.1.19)

We now look at the term $\sigma(x_1, x_2, x_3)$ in the case (3.1.19). We note $x_1 = \alpha$, $x_2 = \beta$ and $x_3 = \varphi$, with $|\alpha| = -1 = |\beta|$ and $\varphi \in \mathcal{H}^2 \subset \mathrm{Hom}(V, V^{\otimes 3})$. Since $[\alpha, \beta]_D = 0$, we have in this case

$$\sigma(\alpha,\beta,\varphi) = \varepsilon\big(\alpha(Q[\beta,\varphi]_D)\big) + \varepsilon\big(\beta(Q[\varphi,\alpha]_D)\big) \tag{3.2.1}$$

Proposition 3.2.4 *We can write $\sigma(\alpha, \beta, \varphi)$ under the form*

$$\sigma(\alpha,\beta,\varphi) = \frac{1}{N-1}\sum_{k,j,l}(\beta_l\alpha_k - \alpha_l\beta_k)\varphi_j^{kjl} = \frac{1}{N-1}\sum_{k,j,l}\alpha_l\beta_k(\varphi_j^{ljk} - \varphi_j^{kjl}) \tag{3.2.2}$$

Proof. Let us compute $Q[\alpha, \varphi]_D$:
we have $[\alpha,\varphi]_{D_j}^{i_0 i_1} = \sum_k \left(\alpha_k\varphi_j^{k i_0 i_1} + \alpha_k\varphi_j^{i_0 k i_1} + \alpha_k\varphi_j^{i_0 i_1 k}\right)$ and

$$S_1([\alpha,\varphi]_D)^{i_1} \overset{3.1.10(ii)}{=} \sum_{k,j}\alpha_k\varphi_j^{kji_1} \tag{3.2.3}$$

because $\varphi \in \mathcal{H}^2$ so $S_2(\varphi) = 0$. We get

$$(\tilde{b}S_1([\alpha,\varphi]_D))_j^{i_0 i_1} = ad(S_1([\alpha,\varphi]_D))_j^{i_0 i_1} = \sum_{k,r}\left(\alpha_k\varphi_r^{kri_0}\delta_j^{i_1} - \delta_j^{i_0}\varphi_r^{kri_1}\right),$$

hence

$$S_1(\tilde{b}S_1([\alpha,\varphi]_D))^{i_1} = \sum_{k,r} \alpha_k \varphi_r^{kri_1} - N\alpha_k\varphi_r^{kri_1} = (1-N)S_1([\alpha,\varphi]_D)^{i_1}.$$

It follows that

$$[\alpha,\varphi]_D + \frac{1}{N-1}\tilde{b}S_1([\alpha,\varphi]_D) = P[\alpha,\varphi]_D = [\alpha,\varphi]_s$$

is in $\operatorname{Ker} S_1 = \mathcal{H}^1$, and so

$$[\alpha,\varphi]_D = [\alpha,\varphi]_D + \frac{1}{N-1}\tilde{b}S_1([\alpha,\varphi]_D) + \tilde{b}\left(\frac{-1}{N-1}S_1([\alpha,\varphi]_D)\right).$$

Hence

$$Q[\alpha,\varphi]_D = -\frac{1}{N-1}S_1([\alpha,\varphi]_D) \overset{(3.2.3)}{=} -\frac{1}{N-1}\sum_{k,j,i_1}\alpha_k\varphi_j^{kji_1}e_{i_1},$$

and the term (3.2.1) becomes

$$\sigma(\alpha,\beta,\varphi) = \frac{1}{N-1}\sum_{k,j,l}(\beta_l\alpha_k - \alpha_l\beta_k)\varphi_j^{kjl} = \frac{1}{N-1}\sum_{k,j,l}\alpha_l\beta_k(\varphi_j^{ljk} - \varphi_j^{kjl}).$$

□

3.2.3 Computations for the coboundary part in the case (3.1.18)

In the case (3.1.18), the ε part vanishes using the Lemma 3.2.3. For the Hochschild complex associated with $\mathcal{T}V$ to be formal, we need that the q part also vanishes, so the arbitrary map $q : \mathcal{H}\wedge\mathcal{H} \to \mathbb{K}$ must satisfy, for $x_1, x_2, x_3 \in \mathcal{H}$ as in the case (3.1.18),

$$0 = -(\delta_{CE}q)(x_1,x_2,x_3) = q([x_1,x_2]_s,x_3) + q([x_2,x_3]_s,x_1) + q([x_3,x_1]_s,x_2). \quad (3.2.4)$$

We have $\mathcal{H}^0 = \operatorname{Hom}(V,V) \cong \mathfrak{gl}_N(\mathbb{K})$. Note $\widetilde{q} = q_{|\mathfrak{gl}_N(\mathbb{K})\wedge\mathfrak{gl}_N(\mathbb{K})}$. As $\sigma(x_1,x_2,x_3) = 0$ for $x_i \in \mathcal{H}^0 = \mathfrak{gl}_N(\mathbb{K})$, we already need that $(\delta_{CE}\widetilde{q}) = 0$ *i.e.* that $\widetilde{q} \in Z^2_{CE}(\mathfrak{gl}_N(\mathbb{K}),\mathbb{K})$ is a 2-cocycle of the scalar Chevalley-Eilenberg cohomology of $\mathfrak{gl}_N(\mathbb{K})$.

Proposition 3.2.5 *The* 2^{nd} *group of the scalar Chevalley-Eilenberg cohomology of* $\mathfrak{gl}_N(\mathbb{K})$ *is trivial, i.e.* $\mathrm{H_{CE}}^2(\mathfrak{gl}_N(\mathbb{K}),\mathbb{K}) = 0$.

Proof. Let $\{E_{ij}\}_{1\leqslant i,j\leqslant N}$ be a base of $\mathfrak{gl}_N(\mathbb{K})$, we have

$$\mathfrak{gl}_N(\mathbb{K}) = \mathfrak{sl}_N(\mathbb{K}) \oplus w\mathbb{K}, \quad \text{where } w = \sum_{k=1}^{N} E_{ii}.$$

Using the Hochschild-Serre Theorem 1.2.2, we have that

$$\mathrm{H_{CE}}^2(\mathfrak{gl}_N(\mathbb{K}),\mathbb{K}) = \bigoplus_{k=0}^{2} \mathrm{H_{CE}}^k(\mathfrak{sl}_N(\mathbb{K}),\mathbb{K}) \otimes \mathrm{H_{CE}}^{2-k}(w\mathbb{K},\mathbb{K}) \cong \mathbf{H}^2_{CE}(w\mathbb{K},\mathbb{K})^{\mathfrak{sl}_N(\mathbb{K})} = 0,$$

because

$$\mathrm{H_{CE}}^k(\mathfrak{sl}_N(\mathbb{K}),\mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } k=0 \\ 0 & \text{if } k=1 \\ 0 & \text{if } k=2 \end{cases}$$

using the theorem of Whitehead.

□

Therefore, the 2-cocylce $\widetilde{q}$ is a 2-coboundary, *i.e.* there exists $p : \mathfrak{gl}_N(\mathbb{K}) \to \mathbb{K}$ such that $\forall\ A,B \in \mathfrak{gl}_N(\mathbb{K}),\ \widetilde{q}(A,B) = -p([A,B])$. There exists $P \in \mathfrak{gl}_N(\mathbb{K})$ such that for $A \in \mathfrak{gl}_N(\mathbb{K}),\ \ p(A) = \sum_{i,j} P_i^j A_j^i = \mathrm{tr}(PA)$.

Now note $q_{-1,1} = q_{|\mathcal{H}^{-1}\wedge\mathcal{H}^1}$, we can write it under the form

$$\begin{array}{rcl} q_{-1,1} : \mathrm{Hom}(V, V^{\otimes 2}) \wedge V^\star & \to & \mathbb{K} \\ \left(\sum_{k,i,j} \psi_k^{ij} e_i \cdot e_j \otimes e^k\right) \wedge \left(\sum_l \alpha_l e^l\right) & \mapsto & \sum_{i,j,k,l} \bar{q}_{ij}^{kl} \psi_k^{ij} \alpha_l \end{array}$$

Proposition 3.2.6 *Under the condition* (3.2.4), *we can write* $\bar{q}_{ij}^{kl}$ *as*

$$\bar{q}_{ij}^{kl} = P_i^k \delta_j^l + P_j^k \delta_i^l - \frac{1}{N} \delta_i^k P_j^l + \nu \delta_i^k \delta_j^l - N\nu \delta_j^k \delta_i^l,$$

where $P = Q + \rho\mathbf{1},\ \mathrm{tr}(Q) = 0$ *and* $\nu \in \mathbb{K}$.

Proof. For $\alpha \in \mathcal{H}^{-1},\ A \in \mathcal{H}^0,\ \psi \in \mathcal{H}^1$, we have

$$[A,\psi]_D{}_k^{ij} = \sum_r \left(A_r^i \psi_k^{rj} + A_r^j \psi_k^{ir} - A_k^r \psi_r^{ij}\right)$$
$$[\psi,\alpha]_D{}_k^i = -[\alpha,\psi]_D{}_k^i = -\sum_r \left(\alpha_r \psi_k^{ri} + \alpha_r \psi_k^{ir}\right),$$

so the equation (3.2.4) reads, with α, A, ψ

$$
\begin{aligned}
0 &= q_{-1,1}(\psi,[\alpha,A]_D) - q_{-1,1}([A,\psi]_D,\alpha) + \operatorname{tr}\left(p([[\psi,\alpha]_D,A])\right)\\
&= \sum_{i,j,k,l,r}\left(\bar{q}^{kl}_{ij}\psi^{ij}_k\alpha_r A^r_l - \bar{q}^{kl}_{ij}A^i_r\psi^{rj}_k\alpha_l - \bar{q}^{kl}_{ij}A^j_r\psi^{ir}_k\alpha_l + \bar{q}^{kl}_{ij}A^r_k\psi^{ij}_r\alpha_l\right)\\
&\quad + \sum_{i,j,k,l,r}\left(P^k_i\sum_s\left(-\alpha_r\psi^{ri}_s A^s_k - \alpha_r\psi^{ir}_s A^s_k + A^i_s\alpha_r\psi^{rs}_k + A^i_s\alpha_r\psi^{sr}_k\right)\right)\\
&= \sum_{i,j,k,l,r}\psi^{ij}_k\alpha_l\left(A^k_r\bar{q}^{rl}_{ij} + A^l_r\bar{q}^{kr}_{ij} - A^r_i\bar{q}^{kl}_{rj} - A^r_j\bar{q}^{kl}_{ir}\right.\\
&\qquad\qquad \left. - \delta^l_i A^k_r P^r_j - \delta^l_j A^k_r P^r_i + \delta^l_i A^r_j P^k_r + \delta^l_j A^r_i P^k_r\right)\\
&= \sum_{i,j,k,l,r}\psi^{ij}_k\alpha_l M^{kl}_{ij}
\end{aligned}
$$

where $M^{kl}_{ij} = [A,\bar{q}]_D{}^{kl}_{ij} + [P,A]_D{}^k_j\delta^l_i + [P,A]_D{}^k_i\delta^l_j$.

Define $p_{-1,1} \in \operatorname{Hom}(V^{\otimes 2}, V^{\otimes 2})$ by

$$
\begin{aligned}
p_{-1,1} &= \sum_{i,j,k,l}\bar{p}^{kl}_{ij}e^i\cdot e^j\otimes e_k\cdot e_l\\
&= \sum_{i,j,k,l}\left(p^k_i\delta^l_j + p^k_j\delta^l_i\right)e^i\cdot e^j\otimes e_k\cdot e_l,
\end{aligned}
$$

so we get $M^{kl}_{ij} = \left([A,q_{-1,1}-p_{-1,1})]_D\right)^{kl}_{ij}$.

We can decompose

$$\psi^{ij}_k = B^i_k v^j \quad \text{with } v\in V,\ B\in\mathfrak{gl}_N(\mathbb{K}) \text{ and } \operatorname{tr}(B)=0,$$

because $\psi\in\mathcal{H}^1 = \operatorname{Ker} S_1$, so $0 = S_1(\psi)^j = \sum_k \psi^{kj}_k$. Then we have $\forall\ \alpha\in V^\star$, $\forall\ v\in V$, $\forall\ B\in\mathfrak{gl}_N(\mathbb{K})$ with $\operatorname{tr}(B)=0$

$$0 = B^i_k v^j\alpha_l M^{kl}_{ij} \Rightarrow B^i_k M^{kl}_{ij} = 0\ \forall\ B \text{ with } \operatorname{tr}(B)=0$$

In particular, for $B = E_{ij}$, we get $M^{kl}_{ij} = \frac{1}{N}\delta^k_i\sum_r M^{rl}_{rj}$, so we have $\forall\ A\in\mathfrak{gl}_N(\mathbb{K})$

$$0 = \left[A,(q_{-1,1}-p_{-1,1}) - \frac{1}{N}\operatorname{tr}(q_{-1,1}-p_{-1,1})\right]_D.$$

Recall the invariant tensor theorem [KMS93, Theorem 24.4 page 214].

Theorem 3.2.7 *Let* $t\in V^{\otimes k}\otimes V^{\star\otimes l}$

$$t = \sum_{i_1,\dots,i_k,j_1,\dots,j_l} t^{i_1\dots i_k}_{j_1\dots j_l} e_{i_1}\cdots e_{i_k}\otimes e^{j_1}\cdots e^{j_l}$$

such that $\forall\ A \in GL(V)$, $[A,t]_D = 0$. *Then*

$$t = \begin{cases} 0 & \text{if } k \neq l \\ \sum_{\sigma\in S_k} a_\sigma \overline{\sigma} & \text{if } k = l \end{cases}$$

where $a_\sigma \in \mathbb{K}$ *and*

$$\begin{aligned}\overline{\sigma} &= \bar{\sigma}^{i_1\dots i_k}_{j_1\dots j_l} e_{i_1}\cdots e_{i_k} \otimes e^{j_1}\cdots e^{j_l} \\ &= \delta^{i_1}_{j\sigma(1)}\cdots\delta^{i_k}_{j\sigma(k)} e_{i_1}\cdots e_{i_k} \otimes e^{j_1}\cdots e^{j_l}.\end{aligned}$$

This theorem gives us

$$\left((q_{-1,1}-p_{-1,1}) - \frac{1}{N}\operatorname{tr}(q_{-1,1}-p_{-1,1})\right)^{kl}_{ij} = \lambda\delta^k_i\delta^l_j + \mu\delta^k_j\delta^l_i,$$

so

$$\bar{q}^{kl}_{ij} = P^k_i\delta^l_j + P^k_j\delta^l_i + \sum_r\left(\frac{1}{N}\bar{q}^{rl}_{rj} - \frac{1}{N}P^r_r\delta^k_i\delta^l_j\right) - \frac{1}{N}\delta^k_i P^l_j + \lambda\delta^k_i\delta^l_j + \mu\delta^k_j\delta^l_i.$$

The contraction $k \leftrightsquigarrow i$ gives $\mu = -N\lambda$ and as before, since $\forall\ \psi \in \mathcal{H}^1 = \operatorname{Ker} S_1$, $\forall\ \alpha \in V^\star$ we have $\sum_{r,j,l}\bar{q}^{rl}_{rj}\psi^{rj}_r\alpha_l = 0$, we get $\sum_r \bar{q}^{rl}_{rj} = 0$.

Finally, replacing P by $Q+\rho\mathbf{1}$, with $\rho\in\mathbb{K}$, $Q\in\mathfrak{gl}_N(\mathbb{K})$, $\operatorname{tr}(Q) = 0$, we have

$$\bar{q}^{kl}_{ij} = Q^k_i\delta^l_j + Q^k_j\delta^l_i - \frac{1}{N}\delta^k_i Q^l_j + \nu\delta^k_i\delta^l_j - N\nu\delta^k_j\delta^l_i \quad \text{with } \nu = \lambda - \frac{\rho}{N}.$$

□

3.2.4 Computations for the coboundary part in the case (3.1.19)

Proposition 3.2.8 *Let* $x_1, x_2, x_3 \in \mathcal{H}$. *In the case* (3.1.19), $(\delta_{CE}q)(x_1,x_2,x_3)$ *vanishes.*

Proof. We note $x_1 = \alpha$, $x_2 = \beta$ and $x_3 = \varphi$, with $|\alpha| = -1 = |\beta|$ and $\varphi \in \mathcal{H}^2 \subset \operatorname{Hom}(V, V^{\otimes 3})$. Since $[\alpha,\beta]_D = 0$, and considering the degrees, the q part left is

$$q_{-1,1}(P[\beta,\varphi]_D,\alpha) - q_{-1,1}([\alpha,\varphi]_D,\beta)$$

In the Section 3.2.2, we have computed that

$$P[\alpha,\varphi]_D = [\alpha,\varphi]_D + \frac{1}{N-1}\tilde{b}S_1([\alpha,\varphi]_D),$$

so we can write

$$\begin{aligned} &q_{-1,1}(P[\beta,\varphi]_D,\alpha) - q_{-1,1}(P[\alpha,\varphi]_D,\beta) \\ = \sum_{k,l,i,j,r} \alpha_k\beta_l \Bigg[\bar{q}^{rk}_{ij} \Bigg(&\varphi_r^{lij} + \varphi_r^{ilj} + \varphi_r^{ijl} + \frac{1}{N-1}\sum_s \left(\delta_r^j \varphi_s^{lsi} - \delta_r^i \varphi_s^{lsj}\right)\Bigg) \\ &- \text{permutation } k \leftrightarrow l \Bigg] \end{aligned}$$

With the Proposition 3.2.6, by replacing $\bar{q}^{rk}_{ij}$ with $Q_i^r\delta_j^k + Q_j^r\delta_i^k - \frac{1}{N}\delta_i^r Q_j^k + \nu\delta_i^r\delta_j^k - N\nu\delta_j^r\delta_i^k$ and expanding, we obtain

$$q_{-1,1}(P[\beta,\varphi]_D,\alpha) - q_{-1,1}(P[\alpha,\varphi]_D,\beta) = 0.$$

□

So in the case (3.1.19), $(\delta_{CE} q)$ vanishes and σ not, so the Hochschild complex of $\mathcal{T}V$ is not formal for V of dimension $N \geqslant 2$.

Suppose now that V is a vectoriel space of any dimension (greater than 2). Choose a subspace $U \subset V$ of finite dimension $N \in \mathbb{N}$. If the Hochschild complex associated with the algebra $\mathcal{T}V$ was formal, we would have a L_∞-morphism between $(\mathfrak{g}, \overline{d})$ and $(\mathcal{G}, \overline{b+D})$, and so $d_3 = 0$. The cohomology class of σ would vanish, and with a good choice of the map q, we would have $\sigma(v_1, v_2, v_3) = 0$ for all $v_i \in \mathfrak{g}_V = \mathbb{K}\mathbf{1} \oplus \mathrm{Hom}(V, \mathcal{T}V)/\mathcal{T}V^+$, $i = 1,2,3$. In particular, for all $u_1, u_2, u_3 \in \mathfrak{g}_U = \mathbb{K}\mathbf{1} \oplus \mathrm{Hom}(U, \mathcal{T}U)/\mathcal{T}U^+ \subset \mathfrak{g}_V$, we would have $\sigma_{|\mathfrak{g}_U}(u_1, u_2, u_3) = 0$ which is not the case because the Hochschild complex of $\mathcal{T}U$ is not formal. Thus, for any vector space V, the Hochschild complex associated with the algebra $\mathcal{T}V$ is not formal.

4

Study of the formality for the Lie algebra $\mathfrak{so}(3)$

Contents

We examine the formality for the Lie algebra $\mathfrak{so}(3)$. We recall its cohomology, and show that the Chevalley-Eilenberg complex retracts on a smaller algebra, which has the same cohomology. We again expose the formality equations and show that they are not satisfied. Computation of a perturbed L_∞ structure gives only a 3-arity component and as for the case of free algebras, we obtain a L_∞-morphism whose components in arity greater than 3 are zero.

4.1 Description of the Lie algebra $\mathfrak{so}(3)$

We consider the Lie algebra $\mathfrak{so}(3)$. It admits for matrix representation the set of 3-dimensional antisymmetric matrices,

$$\mathfrak{so}(3) = \{M \in M_3(\mathbb{K}), \quad {}^tM = -M\} \tag{4.1.1}$$

we can also consider it abstractely, generated by three elements with the following relations

$$\mathfrak{so}(3) =< e_1, e_2, e_3 \mid [e_i, e_{i+1}] = e_{i+2} \qquad i = 1,2,3 \pmod 3 >, \tag{4.1.2}$$

the unmentioned brackets being zero or deduced by antisymmetry.

In that case, generators corresponds to the antisymmetric matrices given by

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the bracket to the matrix commutator : $[x,y] = xy - yx$.

We identify $\mathfrak{so}(3)$ to its dual via the scalar product q, with

$$q(e_i, e_j) := \frac{1}{2}\delta_{ij}$$

on the basis and (abusively using the same name)

$$q(x) = \frac{1}{2}x.x = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

the associated quadratic form.

The Poisson structure associated to the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ is

$$\begin{aligned} P &= \frac{1}{2}\sum_{1\leqslant i,j\leqslant 3} P^{ij}\partial_i \wedge \partial_j \\ &= \frac{1}{2}\big(P^{12}\partial_1 \wedge \partial_2 + P^{13}\partial_1 \wedge \partial_3 + P^{21}\partial_2 \wedge \partial_1 + P^{23}\partial_2 \wedge \partial_3 + P^{31}\partial_3 \wedge \partial_1 + P^{32}\partial_3 \wedge \partial_2\big) \\ &= \frac{1}{2}\big(x_3\partial_1 \wedge \partial_2 - x_2\partial_1 \wedge \partial_3 - x_3\partial_2 \wedge \partial_1 + x_1\partial_2 \wedge \partial_3 + x_2\partial_3 \wedge \partial_1 - x_1\partial_3 \wedge \partial_2\big) \\ &= x_1\partial_2 \wedge \partial_3 + x_2\partial_3 \wedge \partial_1 + x_3\partial_1 \wedge \partial_2, \end{aligned}$$

so $(\mathcal{S}(\mathfrak{so}(3)), \cdot, \{\ ,\ \})$ is a Poisson algebra, with

$$P := \{\ ,\ \} := x_1\partial_2 \wedge \partial_3 + x_2\partial_3 \wedge \partial_1 + x_3\partial_1 \wedge \partial_2 \tag{4.1.3}$$

4.2 Cohomology and subalgebra of the Chevalley-Eilenberg complex

We work with the Lie algebra $\mathfrak{g} := \mathfrak{so}(3)$ and its Chevalley-Eilenberg complex $\mathfrak{A} := \mathrm{C_{CE}}(\mathfrak{g}, \mathcal{S}\,\mathfrak{g}) = \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}(\bigwedge^n \mathfrak{g}, \mathcal{S}\,\mathfrak{g})$.

Since $\mathfrak{so}(3)$ is simple, using Whitehead's theorem, its cohomology $\mathfrak{a}$ with values in $\mathcal{S}\,\mathfrak{g}$ is given by

$$\mathfrak{a}^k := \mathrm{H_{CE}}^k(\mathfrak{g}, \mathcal{S}\,\mathfrak{g}) = \begin{cases} \mathbb{K}[q]\mathbf{1} & \text{if } k = 0, \\ \{0\} & \text{if } k = 1 \text{ or } k = 2, \\ \mathbb{K}[q]\omega & \text{if } k = 3, \end{cases} \tag{4.2.1}$$

where $\omega = \partial_1 \wedge \partial_2 \wedge \partial_3$.

Moreover, the Schouten bracket $[\ ,\ ]_s$ of the graded Lie algebra $\mathfrak{A}[1]$ being of degree 0, it induces on the cohomology $\mathfrak{a}[1]$ a zero bracket $[\ ,\ ]'_s \equiv 0$; $\mathfrak{a}[1]$ is thus an abelian Lie algebra.

$$\begin{array}{cccccccc} \mathfrak{a}[1] = & \mathbb{K}[q]\mathbf{1} & \oplus & \{0\} & \oplus & \{0\} & \oplus & \mathbb{K}[q]\omega \\ \text{Degree} & -1 & & 0 & & 1 & & 2 \end{array}$$

Indeed, if $f, g \in \mathbb{K}[q]$, then $[f,g]'_s = 0$ (degree -2), $[f\omega, g\omega]'_s = 0$ (degree 4) and $[f, g\omega]'_s = 0$ because $\mathfrak{a}[1]^1 = \{0\}$.

More generally, we compute the Schouten brackets between these four components at the level of the complex $\mathfrak{A}[1]$.

Proposition 4.2.1 *The elements*

unit $\mathbf{1} \in \mathcal{S}\,\mathfrak{g}$, *unit for the graded commutative multiplication* $\wedge$,

Euler field $E := x_1\partial_1 + x_2\partial_2 + x_3\partial_3$ *“trace” derivation,*

Poisson structure $P := x_1\partial_2 \wedge \partial_3 + x_2\partial_3 \wedge \partial_1 + x_3\partial_1 \wedge \partial_2$,

basis trivector $\omega := \partial_1 \wedge \partial_2 \wedge \partial_3$

generate (on $\mathbb{K}[q]$*) a subalgebra* $\mathfrak{A}_{\mathrm{red}}[1] =< \mathbf{1}, E, P, \omega >$ *of the graded Lie algebra* $(\mathfrak{A}[1], [\ ,\ ]_s)$.

Lemma 4.2.2 *Let* $f = \alpha(q) \in \mathbb{K}[q]$. *We have the following identities.*

(i) $\partial_i f = \partial_i \alpha(q) = x_i \alpha'(q)$ *for* $1 \leqslant i \leqslant 3$,

(ii) $[E, \alpha(q)\mathbf{1}]_s = 2q\alpha'(q)$,

(iii) $[P, \alpha(q)\mathbf{1}]_s = \delta_{CE}(\alpha(q)\mathbf{1}) = 0$,

(iv) $[P, E]_s = P$,

(v) $[\omega, E]_s = 3\omega$,

(vi) $P \wedge E = 2q\omega$.

Proof.

(i)

$$\partial_i f = \partial_i \alpha(q) = \frac{\partial \alpha}{\partial q}\frac{\partial q}{\partial x_i} = \alpha'(q)\frac{\partial}{\partial x_i}\left(\frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)\right) = x_i\alpha'(q).$$

(ii)

$$E(\alpha(q)) = \sum_{i=1}^{3} x_i\partial_i(\alpha(q)) = \sum_{i=1}^{3} x_i x_i \alpha'(q) = 2q\alpha'(q).$$

(iii) We have

$$\begin{aligned}[x_1\partial_2\wedge\partial_3, \alpha(q)\mathbf{1}]_s &= -(-1)^{1\times(-1)}[\alpha(q)\mathbf{1}, x_1\partial_2\wedge\partial_3]_s\\ &= [\alpha(q)\mathbf{1}, x_1\partial_2]_s\wedge\partial_3 + (-1)^{-1\times 0}x_1\partial_2\wedge[\alpha(q)\mathbf{1},\partial_3]_s\\ &= -x_1\partial_2(\alpha(q))\wedge\partial_3 - x_1\partial_2\wedge\partial_3(\alpha(q))\\ &= -x_1\alpha'(q)(x_2\partial_3 - x_3\partial_2)\end{aligned}$$

and similarly

$$\begin{aligned}[x_2\partial_3\wedge\partial_1, \alpha(q)]_s &= -x_2\alpha'(q)(x_3\partial_1 - x_1\partial_3),\\ [x_3\partial_1\wedge\partial_2, \alpha(q)]_s &= -x_3\alpha'(q)(x_1\partial_2 - x_2\partial_1),\end{aligned}$$

thus

$$[P,\alpha(q)\mathbf{1}]_s = -\alpha'(q)(x_1x_2\partial_3 - x_1x_3\partial_2 + x_2x_3\partial_1 - x_2x_1\partial_3 + x_3x_1\partial_2 - x_3x_2\partial_1) = 0.$$

(iv)

$$\begin{aligned}[P,E]_s &= -[E,P]_s\\ &= -[x_1\partial_1 + x_2\partial_2 + x_3\partial_3, x_1\partial_2\wedge\partial_3 + x_2\partial_3\wedge\partial_1 + x_3\partial_1\wedge\partial_2]_s.\end{aligned}$$

For $1 \leqslant i \leqslant 3$,

$$\begin{aligned}[x_i\partial_i, x_1\partial_2\wedge\partial_3]_s &= [x_i\partial_i, x_1\partial_2]_s\wedge\partial_3 + (-1)^{0\times 1}x_1\partial_2\wedge[x_i\partial_i,\partial_3]_s\\ &= (x_i\delta_{i1}\partial_2 - x_1\delta_{2i}\partial_i)\wedge\partial_3 - x_1\partial_2\wedge\delta_{3i}\partial_i\end{aligned}$$

and similarly

$$\begin{aligned}[x_i\partial_i, x_2\partial_3\wedge\partial_1]_s &= (x_i\delta_{i2}\partial_3 - x_2\delta_{3i}\partial_i)\wedge\partial_1 - x_2\partial_3\wedge\delta_{1i}\partial_i,\\ [x_i\partial_i, x_3\partial_1\wedge\partial_2]_s &= (x_i\delta_{i3}\partial_1 - x_3\delta_{1i}\partial_i)\wedge\partial_2 - x_3\partial_1\wedge\delta_{2i}\partial_i.\end{aligned}$$

Thus,

$$\begin{aligned}[E,P]_s = &\; x_1\partial_2\wedge\partial_3 - x_2\partial_3\wedge\partial_1 - x_3\partial_1\wedge\partial_2\\ &- x_1\partial_2\wedge\partial_3 + x_2\partial_3\wedge\partial_1 - x_3\partial_1\wedge\partial_2\\ &- x_1\partial_2\wedge\partial_3 - x_2\partial_3\wedge\partial_1 + x_3\partial_1\wedge\partial_2\\ = &\; -2P + P = -P,\end{aligned}$$

and so $[P,E]_s = P$.

(v)

$$\begin{aligned}[\omega, E]_s &= -[x_1\partial_1 + x_2\partial_2 + x_3\partial_3, \partial_1 \wedge \partial_2 \wedge \partial_3]_s \\ &= -[E, \partial_1]_s \wedge \partial_2 \wedge \partial_3 - \partial_1 \wedge [E, \partial_2]_s \wedge \partial_3 - \partial_1 \wedge \partial_2 \wedge [E, \partial_3]_s \\ &= 3\omega.\end{aligned}$$

(vi)

$$\begin{aligned}P \wedge E &= (x_1\partial_2 \wedge \partial_3 + x_2\partial_3 \wedge \partial_1 + x_3\partial_1 \wedge \partial_2) \wedge (x_1\partial_1 + x_2\partial_2 + x_3\partial_3) \\ &= x_1^2\partial_2 \wedge \partial_3 \wedge \partial_1 + x_2^2\partial_3 \wedge \partial_1 \wedge \partial_2 + x_3^2\partial_1 \wedge \partial_2 \wedge \partial_3 \\ &= 2q\omega.\end{aligned}$$

□

Proof of the Proposition. We have to show the stability of the reduced algebra by the Schouten bracket $[\ ,\ ]_s$. Let $\alpha, \beta \in \mathbb{K}[q]$, using the identities of the preceding lemma, we obtain

$$[\alpha\mathbf{1}, \beta\mathbf{1}]_s = 0,$$

$$[\alpha E, \beta\mathbf{1}]_s = \alpha E(\beta)\mathbf{1} = 2q\alpha\beta'\mathbf{1},$$

$$[\alpha P, \beta\mathbf{1}]_s = \alpha[P, \beta\mathbf{1}]_s = \alpha\delta_{CE}(\beta\mathbf{1}) = 0,$$

$$\begin{aligned}[\alpha\omega, \beta\mathbf{1}]_s &= \alpha[\partial_1 \wedge \partial_2 \wedge \partial_3, \beta\mathbf{1}]_s \\ &= \alpha\big([\partial_1, \beta\mathbf{1}] \wedge \partial_2 \wedge \partial_3 - \partial_1 \wedge [\partial_2, \beta\mathbf{1}] \wedge \partial_3 + \partial_1 \wedge \partial_2 \wedge [\partial_3, \beta\mathbf{1}]\big) \\ &= \alpha\big(\beta' x_1\partial_2 \wedge \partial_3 - \beta' x_2\partial_1 \wedge \partial_3 + \beta' x_3\partial_1 \wedge \partial_2\big) \\ &= \alpha\beta' P,\end{aligned}$$

$$[\alpha E, \beta E]_s = \alpha E(\beta)E - \beta E(\alpha)E = 2q(\alpha\beta' - \alpha'\beta)E,$$

$$\begin{aligned}[\alpha P, \beta E]_s &= \beta[\alpha P, E]_s = -\beta[E, \alpha P]_s \\ &= -\beta E(\alpha)P - \beta\alpha[E, P]_s \\ &= \beta(\alpha - 2q\alpha')P,\end{aligned}$$

$$\begin{aligned}[\alpha\omega, \beta E]_s &= \alpha[\omega, \beta E]_s + [\alpha\mathbf{1}, \beta E]_s \wedge \omega \\ &= \alpha\big([\omega, \beta\mathbf{1}] \wedge E + \beta[\omega, E]\big) - 2q\beta\alpha'\omega \\ &= \alpha\big(\beta' P \wedge E + 3\beta\omega\big) - 2q\alpha'\beta\omega \\ &= \alpha\big(2q\beta'\omega + 3\beta\omega\big) - 2q\alpha'\beta\omega \\ &= \big(2q(\alpha\beta' - \alpha'\beta) + 3\alpha\beta\big)\omega,\end{aligned}$$

and

$$[\alpha P, \beta P]_s = 0, \qquad [\alpha\omega, \beta P]_s = 0 \qquad [\alpha\omega, \beta\omega]_s = 0,$$

the last two equalities for degree reasons. Thus, the various brackets give multiples (in $\mathbb{K}[q]$) of the elements E, P, ω. □

This reduced algebra can also be written

$$\mathfrak{A}_{\mathrm{red}}[1] = \mathbb{K}[q]\mathbf{1} \oplus \mathbb{K}[q]E \oplus \mathbb{K}[q]P \oplus \mathbb{K}[q]\omega = H \oplus W, \tag{4.2.2}$$

where $H := \mathbb{K}[q]\mathbf{1} \oplus \mathbb{K}[q]\omega = \mathfrak{a}[1]$ is the cohomology and $W = \mathbb{K}[q]E \oplus \mathbb{K}[q]\omega$ its supplement in $\mathfrak{A}_{\mathrm{red}}[1]$.

4.3 Deformation retract of the complex

We will compute the cohomology of the subcomplex $(\mathfrak{A}_{\mathrm{red}}, \delta_{CE})$, we obtain the same cohomology than for the entire Chevalley-Eilenberg complex $\mathfrak{A} = \mathrm{C}_{\mathrm{CE}}(\mathfrak{g}, \mathcal{S}\,\mathfrak{g})$.

Theorem 4.3.1 *The cohomology of the subcomplex $(\mathfrak{A}_{\mathrm{red}}, \delta_{CE})$ is given by*

$$\mathfrak{a}^k_{\mathrm{red}} := \begin{cases} \mathbb{K}[q]\mathbf{1} & \text{if } k = 0, \\ \{0\} & \text{if } k = 1 \text{ or } k = 2, \\ \mathbb{K}[q]\omega & \text{if } k = 3. \end{cases} \tag{4.3.1}$$

Proof. Let $\alpha \in \mathbb{K}$, we use the Lemma 4.2.2 to obtain the following results.

$\delta_{CE}(\alpha\mathbf{1}) = 0$, so $Z^0\mathfrak{A}_{\mathrm{red}} = \mathfrak{A}^0_{\mathrm{red}} = \mathbb{K}[q]$ and $\mathfrak{a}^0_{\mathrm{red}} = \mathbb{K}[q]$ since there are no coboundaries in degree 0.

$\delta_{CE}(\alpha E) = [P, \alpha E]_s = \alpha P$ so αE is a cocycle if and only if $\alpha = 0$, therefore $Z^1\mathfrak{A}_{\mathrm{red}} = \{0\}$ and it follows that $\mathfrak{a}^1_{\mathrm{red}} = \{0\}$.

$\delta_{CE}(\alpha P) = 0$, so $Z^2\mathfrak{A}_{\mathrm{red}} = \mathfrak{A}^2_{\mathrm{red}}$. Moreover, αP is a coboundary, because $\alpha P = \delta_{CE}(\alpha E)$, so $B^2\mathfrak{A}_{\mathrm{red}} = Z^2\mathfrak{A}_{\mathrm{red}}$ and thus $\mathfrak{a}^2_{\mathrm{red}} = \{0\}$.

$\delta_{CE}(\alpha\omega) = [P, \alpha\omega]_s = 0$, so $Z^3\mathfrak{A}_{\mathrm{red}} = \mathfrak{A}^3_{\mathrm{red}}$ and $B^3\mathfrak{A}_{\mathrm{red}} = \{0\}$ cause $\delta(\gamma P) = 0$ for all $\gamma \in \mathbb{K}[q]$, so $\mathfrak{a}^3_{\mathrm{red}} = \mathbb{K}[q]\omega$.

□

Thus, the Chevalley-Eilenberg complex $\mathfrak{A}$ retracts by deformation on the reduced complex $\mathfrak{A}_{\mathrm{red}}$, which has the same cohomology. Note $i : \mathfrak{A}_{\mathrm{red}} \to \mathfrak{A}$ and $\mathrm{pr}_{\mathfrak{A}_{\mathrm{red}}} : \mathfrak{A} \to \mathfrak{A}_{\mathrm{red}}$ the canonical inclusion and projection.

Theorem 4.3.2 *The cochain complex $(\mathfrak{A}_{\mathrm{red}}, \delta_{CE})$ is a deformation retract of $(\mathfrak{A}, \delta_{CE})$ i.e. there is a homotopy h of degree -1 between $id_{\mathfrak{A}}$ et $\mathrm{pr}_{\mathfrak{A}_{\mathrm{red}}}$,*

$$(\mathfrak{A}_{\mathrm{red}}, \delta_{CE}|_{\mathfrak{A}_{\mathrm{red}}}) \underset{\mathrm{pr}_{\mathfrak{A}_{\mathrm{red}}}}{\overset{i}{\rightleftarrows}} (\mathfrak{A}, \delta_{CE}) \quad \circlearrowright h\,, \tag{4.3.2}$$

map satisfying $\delta h + h\delta = id_{\mathfrak{A}} - \mathrm{pr}_{\mathfrak{A}_{\mathrm{red}}}$.

Proof. Noting $h_k : \mathfrak{A}^k \to \mathfrak{A}^{k-1}$, the map h defined by

$$h = \begin{cases} h_0 = 0 \\ h_1 = 0 \\ h_2(\alpha P) = \alpha E \\ h_3 = 0 \end{cases}$$

fits, moreover, it verifies $h\delta_{CE} h = h$. □

4.4 Computation of the L_∞ structure

We will study the formality equations (2.3.8), working on $\mathfrak{A}[2]$. For the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$, we have $D = D_2$, $[\ ,\ ]'_s \equiv 0$ so $d_2 = 0$, and a section going back from the cohomology to the cocycles is given by $\varphi_1 = \mathrm{inc}$ the canonical inclusion of the components of the cohomology in the Chevalley-Eilenberg complex, we will omit it in the computations. For $y_1, \dots, y_{k+1} \in \mathfrak{a}[2]$, we obtain at the first levels the following equations.

Level $k = 0$ $\;0 = \delta_{CE}\,\mathrm{inc}$,

Level $k = 1$

$$0 = \delta_{CE}\varphi_2(y_1, y_2) + D(y_1, y_2), \tag{4.4.1}$$

Level $k = 2$

$$\begin{aligned} d_3(y_1, y_2, y_3) = \delta_{CE}\varphi_3(y_1, y_2, y_3) &+ D(y_1, \varphi_2(y_2, y_3)) \\ &+ (-1)^{|y_2||y_3|} D(y_2, \varphi_2(y_1, y_3)) \\ &+ (-1)^{|y_3|(|y_1|+|y_2|)} D(y_3, \varphi_2(y_1, y_2)) \end{aligned} \tag{4.4.2}$$

Let $\alpha, \beta \in \mathbb{K}[q]$. Since φ_2 is of degree 0, it is sufficient to consider at the first order $k = 1$ elements $y_1 = \alpha\,\mathbf{1}$ and $y_2 = \beta\omega$. Writing $\varphi_2(\alpha\,\mathbf{1}, \beta\omega) = \gamma E$, with $\gamma \in \mathbb{K}[q]$, we have $\delta_{CE}\varphi_2(\alpha\,\mathbf{1}, \beta\omega) = [P, \gamma E]_s = \gamma P$ so

$$\delta_{CE}\varphi_2(\alpha\,\mathbf{1}, \beta\omega) + D(\alpha\,\mathbf{1}, \beta\omega) = 0 \Rightarrow \gamma P = \alpha' \beta P.$$

Thus we can define the symmetric graded map $\varphi_2 : \mathfrak{a}[2] \to \mathfrak{A}[2]$ by

$$\begin{aligned} \varphi_2(\alpha\,\mathbf{1}, \beta\,\mathbf{1}) &= 0 \\ \varphi_2(\alpha\,\mathbf{1}, \beta\omega) &= \alpha'\beta E \\ \varphi_2(\alpha\omega, \beta\omega) &= 0 \end{aligned}$$

More generally, $\varphi_n : \mathfrak{a}[2]^{\otimes n} \to \mathfrak{A}[2]$ being of degree 0, for $i_1, \dots, i_n \in \{-2, 1\}$, $\varphi_n(\mathfrak{a}[2]^{i_1}, \dots, \mathfrak{a}[2]^{i_n}) \subset \mathfrak{A}[2]^{i_1+\dots+i_n}$ so, choosing p elements of degree -2 and $n-p$ elements of degree 1, the image of φ_n is in the component of degree $1(n-p) - 2p = n - 3p$. Restricting to the subalgebra $\mathfrak{A}_{\text{red}}[2]$, we have $-2 \leqslant n - 3p \leqslant 1 \Leftrightarrow n - 1 \leqslant 3p \leqslant n + 2$.

At the level $k = 2$, the map φ_3 takes as argument $p = 1$ ($2 \leqslant 3p \leqslant 5$) element of degree -2, noted $y_1 = \alpha\,\mathbf{1}$ and the other elements of degrees 1, $y_2 = \beta_2\omega, y_3 = \beta_3\omega$, with $\alpha, \beta_2, \beta_3 \in \mathbb{K}[q]$.

The right-hand side of equation (4.4.2) then writes, using identities of Lemma 4.2.2

$$\begin{aligned}
&\delta_{CE}(\varphi_3(\alpha\,\mathbf{1}, \beta_2\omega, \beta_3\omega)) \\
&\quad + (-1)^1(-1)^{1\times(-2)}[\beta_2\omega, \varphi_2(\alpha\,\mathbf{1}, \beta_3\omega)]_s + (-1)^1(-1)^{1\times(-2+1)}[\beta_3\omega, \varphi_2(\alpha\,\mathbf{1}, \beta_2\omega)]_s \\
&= \delta_{CE}(\varphi_3(\alpha\,\mathbf{1}, \beta_2\omega, \beta_3\omega)) - [\beta_2\omega, \alpha'\beta_3 E]_s + [\beta_3\omega, \alpha'\beta_2 E]_s \\
&= \delta_{CE}(\varphi_3(\alpha\,\mathbf{1}, \beta_2\omega, \beta_3\omega)) + 4q\alpha'\left(\beta_2'\beta_3 - \beta_2\beta_3'\right)\omega
\end{aligned}$$

and the ω multiple is not a coboundary. This shows that there is no formality in this case.

Theorem 4.4.1 *The Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{so}(3)$ is not formal.*

Proof. In the right-hand side of equation (4.4.2), the multiple (not zero) of ω cannot be written $\delta_{CE}(\alpha P)$ because $[P, \alpha P]_s = 0$. Thus, we cannot choose a map φ_3 which would cancel the ω term. If we choose in (4.4.1) another map $\varphi_2' = \varphi_2 + \chi_2$, then $\delta_{CE}(\chi_2) = 0$. Since $\chi_2 \in Z^1(\mathfrak{g}, \mathcal{S}\,\mathfrak{g}) = \{0\}$, it is a coboundary $\chi_2 = \delta_{CE}(f)$, with $f \in \mathrm{C_{CE}}^0(\mathfrak{g}, \mathcal{S}\,\mathfrak{g}) = \mathcal{S}\,\mathfrak{g}$. But then

$$\begin{aligned}
D(y_i, \chi_2(y_j, y_k)) &= (-1)^{|y_i|}[y_i, [P, f(y_j, y_k)]_s]_s \\
&= -[P, [y_i, f(y_j, y_k)]_s]_s \\
&= (-1)^{|y_i|+1}\delta_{CE} D(y_i, f(y_j, y_k))
\end{aligned}$$

since $[P, y_i]_s = 0$, and this only modifies the term φ_3 in the equation (4.4.2), so we cannot cancel d_3. □

In order to have a L_∞-morphism, the differential on the cohomology must have higher order components. Order 3 suffice.

Theorem 4.4.2 *There is a L_∞ structure $(\mathfrak{a}[1], \overline{d})$ induced by $d = d_3$ and a L_∞-morphism induced by $\varphi = \varphi_1 + \varphi_2$ noted $\overline{\varphi} : \mathcal{S}(\mathfrak{a}[2]) \to \mathcal{S}(\mathfrak{A}[2])$ such that $\overline{\delta_{CE} + D} \circ \overline{\varphi} = \overline{\varphi} \circ \overline{d}$.*

Proof. Using Theorem 2.5.7 on the contraction

$$\mathfrak{a}[1] \underset{p}{\overset{i}{\rightleftarrows}} (\mathfrak{A}[1], \delta_{CE}) \qquad \circlearrowright h\,,$$

we have

$$\overline{d} = \overline{\psi_1} \circ \sum_{r\in\mathbb{N}} (-\overline{D_2} \circ \eta)^r \circ \overline{D_2} \circ \overline{\varphi_1} = \overline{psi_1} \circ \overline{D_2} \circ \sum_{r\in\mathbb{N}} (-\eta \circ \overline{D_2})^r \circ \overline{\varphi_1}$$

$$\overline{\varphi} = \sum_{r\in\mathbb{N}} (-\eta \circ \overline{D_2})^r \overline{\varphi_1}$$

with $\psi_1 = p[1]$ et $\varphi_1 = i[1]$.

Let $T_3 := \alpha\,\mathbf{1} \bullet \beta_2\omega \bullet \beta_3\omega \in \mathcal{S}(V[1])$. Since $\overline{\varphi_1} = id_{\mathcal{S}(H[1])}$, we have

$$\varphi_3 = \mathrm{pr}_{V[1]} \circ \overline{\varphi}|_3 = \mathrm{pr}_{V[1]} \circ \sum_{n\in\mathbb{N}} (-\eta \circ \overline{D_2})^n$$

and

$$\varphi_3(T_3) = \mathrm{pr}_{V[1]}(\alpha\,\mathbf{1} \bullet \beta_2\omega \bullet \beta_3 + \alpha'\beta_2 E \bullet \beta_3\omega - \alpha'\beta_3 E \bullet \beta_2\omega) = 0,$$

because $D_2(\alpha\,\mathbf{1}, \beta_i\omega) = -\alpha'\beta_i P$ and $\overline{h}(\gamma P) = \gamma E$, the power $(-\eta \circ \overline{D_2})^n$ vanish for $n \geqslant 2$ because D_2 gives multiples of $E \bullet \omega$ and η (defined with $\overline{h}$) is not zero only on words in P ; finally the projection on $V[1] = \mathcal{S}^1(V[1])$ vanish since the considered elements are words of length greater than 1.

Using that $D_2(\alpha E, \beta\omega) = [\beta\omega, \alpha E]_s$, computation of the term d_3 gives again the same expression as before.

$$\begin{aligned} d_3(T_3) &= \psi_1 \circ \overline{D_2} \circ \overline{\varphi}|_3(T_3) \\ &= \psi_1(-\alpha'\beta_2 P \bullet \beta_3\omega + \alpha'\beta_3 P \bullet \beta_2\omega + D_2(\alpha'\beta_2 E, \beta_3\omega) - D_2(\alpha'\beta_3 E, \beta_2\omega)) \\ &= 4q\alpha'(\beta_2'\beta_3 - \beta_2\beta_3')\omega \end{aligned}$$

More generally, the iterated compositions $(-\eta\circ\overline{D_2})^n$ on $T = \alpha_1\,\mathbf{1} \bullet \cdots \bullet \alpha_k\,\mathbf{1} \bullet \beta_1\omega \bullet \cdots \bullet \beta_l\omega \in \mathcal{S}(H[1]) \otimes \mathcal{S}(W[1])$ with $k \geqslant 2$ eventually vanish because of the values taken by D_2 and the definition of the chain homotopy h involved in η. Thus, $\varphi_p(T) = 0$ for $p \geqslant 3$ and $d_p(T) = \psi_1 \circ \overline{D_2} \circ \overline{\varphi}|_p(T) = 0$ for $p \geqslant 4$ with $k \geqslant 2$. □

Part II

Hom-algebraic structures

Table of Contents

Introduction of the second part

In this second part, we present various Hom-algebraic structures and ways to deform them. The main characteristic of Hom-algebras and Hom-cogalebras is the fact that classical identities are deformed by an endomorphism. The main part of this work was prepublished in the article [BEM12].

We first recall (Chapter 5) the definitions of Hom-associative and Hom-Lie algebras, as well as the corresponding dual structures ; some examples are given.

Duality and twisting are two principles to construct more Hom-structures of the same type than given ones. We explain under which conditions this is possible, and we compute such twist morphisms on some examples.

Hom-Poisson algebras were introduced in [MS10b], where they emerged naturally in the study of 1-parameter formal deformation of commutative Hom-associative algebras. This structure was then studied in [Yau10b]. It is shown that the they are closed under twisting by suitable self maps and under tensor product. Moreover, it is shown that (de)polarized Hom-Poisson algebras are equivalent to admissible Hom-Poisson algebras, each of which only has one binary operation.

The twisting principle is also true, we use it to deform the Sklyanin algebra in Example 6.1.5. Section 6.1.4 shows how to build Hom-Poisson algebras starting from Hom-Lie algebras in finite dimension, and introduce such constructions on 3-dimensional examples. Section 6.2 presents the notion of flexible Hom-algebra, more general that Hom-associative algebra, and shows its links with one operation Hom-Poisson algebras. Hopf Hom-coPoisson structures are also defined in Section Section 6.3. Theorem 6.3.5 establish a correspondence between enveloping algebra of Hom-Lie algebra endowed with a Hom-coPoisson structure and Hom-Lie bialgebras. Duality between coPoisson Hopf algebras and Poisson-Hopf algebras is extended to the Hom setting in Theorem 6.3.8.

In Chapter 7, we give the definitions of formal deformation which has been extended for Hom-associatives algebras, Hom-co-algebras and Hom-bialgebras. We finally study the case of the Moyal-Weyl $\star$-product corresponding to the quantization of the phases space Poisson bracket. Proposition 7.2.6 shows that the twisting morphisms of the (2-dimensional) phases space Poisson bracket are of jacobian 1. To obtain twisting morphisms of the $\star$-product, we quantize Poisson automorphisms by modifying a differential equation.

5

Twisting usual structures: Hom-algebras and Hom-coalgebras

Contents

We briefly present definitions and properties of usual Hom-structures; we will show some examples and recall the twisting principle allowing oneself to deform a classical structure into a Hom-structure by means of an algebra morphism. Duality between algebras and coalgebras is also holds for Hom-structures.

5.1 Hom-associative algebras and Hom-Lie algebras

Throughout the next sections, $\mathbb{K}$ denotes a field of characteristic 0 and A a $\mathbb{K}$-module. In the sequel we denote by σ the linear map $\sigma : A^{\otimes 3} \to A^{\otimes 3}$, defined as $\sigma(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_3 \otimes x_1$ and by τ_{ij} the linear maps $\tau : A^{\otimes n} \to A^{\otimes n}$ where $\tau_{ij}(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n) = (x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n)$.

We mean by a Hom-algebra a triple (A, μ, α) in which $\mu : A^{\otimes 2} \to A$ is a linear map and $\alpha : A \to A$ is a linear self map. The linear map $\mu^{op} : A^{\otimes 2} \to A$ denotes the opposite map, *i.e.* $\mu^{op} = \mu \circ \tau_{12}$. A Hom-coalgebra is a triple (A, Δ, α) in which $\Delta : A \to A^{\otimes 2}$ is a linear map and $\alpha : A \to A$ is a linear self map. The linear map $\Delta^{op} : A \to A^{\otimes 2}$ denotes the opposite map, *i.e.* $\Delta^{op} = \tau_{12} \circ \Delta$. For a linear self-map $\alpha : A \to A$, we denote by α^n the n-fold composition of n copies of α, with $\alpha^0 = id$. A Hom-algebra (A, μ, α) (resp. a Hom-coalgebra (A, Δ, α)) is said to be *multiplicative* if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ (resp. $\alpha^{\otimes 2} \circ \Delta = \Delta \circ \alpha$) . The Hom-algebra is called *commutative* if $\mu = \mu^{op}$ and the Hom-coalgebra is called *cocommutative* if $\Delta = \Delta^{op}$.

Classical algebras or coalgebras are also regarded as a Hom-algebras or Hom-coalgebras with identity twisting map. Given a Hom-algebra (A, μ, α), we often use the abbreviation $\mu(x, y) = xy$ for $x, y \in A$. Likewise, for a Hom-coalgebra (A, Δ, α), we will use Sweedler's notation $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ but often omit the symbol of summation. When the Hom-algebra (resp. Hom-coalgebra) is multiplicative, we also say that α is multiplicative for μ (resp. Δ).

5.1.1 Definitions

We recall the definitions of Hom-Lie algebras and Hom-associative algebras.

Definition 5.1.1 Let (A, μ, α) be a Hom-algebra.

1. The *Hom-associator* $\mathfrak{as}_{\mu,\alpha} : A^{\otimes 3} \to A$ is defined as

$$\mathfrak{as}_{\mu,\alpha} = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu). \tag{5.1.1}$$

2. The Hom-algebra A is called a *Hom-associative algebra* if it satisfies the Hom-associative identity

$$\mathfrak{as}_{\mu,\alpha} = 0. \tag{5.1.2}$$

3. A Hom–associative algebra A is called *unital* if there exists a linear map $\eta : \mathbb{K} \to A$ such that

$$\mu \circ (id_A \otimes \eta) = \mu \circ (\eta \otimes id_A) = \alpha. \qquad (5.1.3)$$

The unit element is $1_A = \eta(1)$.

4. The *Hom-Jacobian* $J_{\mu,\alpha} : A^{\otimes 3} \to A$ is defined as

$$J_{\mu,\alpha} = \mu \circ (\alpha \otimes \mu) \circ (id + \sigma + \sigma^2). \qquad (5.1.4)$$

5. The Hom-algebra A is called a *Hom-Lie algebra* if it satisfies $\mu + \mu^{op} = 0$ and the Hom-Jacobi identity

$$J_{\mu,\alpha} = 0. \qquad (5.1.5)$$

We recover the usual definitions of the associator, an associative algebra, the Jacobian, and a Lie algebra when the twisting map α is the identity map. In terms of elements $x, y, z \in A$, the Hom-associator and the Hom-Jacobian are

$$\begin{aligned} \mathfrak{as}_{\mu,\alpha}(x,y,z) &= (xy)\alpha(z) - \alpha(x)(yz), \\ J_{\mu,\alpha}(x,y,z) &= \circlearrowleft_{x,y,z} [\alpha(x),[y,z]], \end{aligned}$$

where the bracket denotes the product and $\circlearrowleft_{x,y,z}$ denotes the cyclic sum on x, y, z.

Let $A = (A, \mu, \alpha)$ and $A' = (A', \mu', \alpha')$ be two Hom-algebras. A linear map $f : A \to A'$ is a *morphism of Hom-algebras* if

$$\mu' \circ (f \otimes f) = f \circ \mu \quad \text{and} \qquad f \circ \alpha = \alpha' \circ f.$$

It is said to be a *weak morphism* if holds only the first condition.

Proposition 5.1.2 *[MS08, Proposition 1.6] To any Hom-associative algebra (A, μ, α), one may associate a Hom-Lie algebra defined for all $x, y, z \in A$ by the bracket $[x, y] = \mu(x, y) - \mu(y, x)$.*

Proof. The bracket is obviously skewsymmetric and with a direct computation we have

$$\begin{aligned} & [\alpha(x),[y,z]] + [\alpha(y),[z,x]] + [\alpha(z),[x,y]] \\ = \quad & \mu(\alpha(x),\mu(y,z)) - \mu(\alpha(x),\mu(z,y)) - \mu(\mu(y,z),\alpha(x)) + \mu(\mu(z,y),\alpha(x)) \\ & + \mu(\alpha(y),\mu(z,x)) - \mu(\alpha(y),\mu(x,z)) - \mu(\mu(z,x),\alpha(y)) + \mu(\mu(x,z),\alpha(y)) \\ & + \mu(\alpha(z),\mu(x,y)) - \mu(\alpha(z),\mu(y,x)) - \mu(\mu(x,y),\alpha(z)) + \mu(\mu(y,x),\alpha(z)) \\ = \; & 0. \end{aligned}$$

□

Example 5.1.3 Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over $\mathbb{K}$. The following multiplication μ and linear map α on $A = \mathbb{K}^3$ define a Hom-associative algebra over $\mathbb{K}$.

$$\mu(x_1,x_1) = ax_1, \qquad \mu(x_2,x_2) = ax_2,$$
$$\mu(x_1,x_2) = \mu(x_2,x_1) = ax_2, \qquad \mu(x_2,x_3) = bx_3,$$
$$\mu(x_1,x_3) = \mu(x_3,x_1) = bx_3, \qquad \mu(x_3,x_2) = \mu(x_3,x_3) = 0,$$

$$\alpha(x_1) = ax_1, \quad \alpha(x_2) = ax_2, \quad \alpha(x_3) = bx_3$$

where a, b are parameters in $\mathbb{K}$. This algebra is not associative when $a \neq b$ and $b \neq 0$, since

$$\mu(\mu(x_1,x_1),x_3)) - \mu(x_1,\mu(x_1,x_3)) = (a-b)bx_3.$$

Example 5.1.4 (Jackson $\mathfrak{sl}(2)$) The Jackson $\mathfrak{sl}(2)$ is a q-deformation of the classical Lie algebra $\mathfrak{sl}(2)$. It carries a Hom-Lie algebra structure but not a Lie algebra structure by using Jackson derivations. It is defined with respect to a basis $\{e, f, h\}$ by the brackets and a linear (non multiplicative) map α such that

$$[h,e] = 2e, \qquad \alpha(e) = qe,$$
$$[h,f] = -2qf, \qquad \alpha(f) = q^2 f,$$
$$[e,f] = \frac{q+1}{2}h, \qquad \alpha(h) = qh,$$

where q is a parameter in $\mathbb{K}$. If $q = 1$ we recover the classical $\mathfrak{sl}(2)$.

5.1.2 Twisting principle

The following proposition gives an easy way to twist classical structures into Hom-structures.

Theorem 5.1.5 ([Yau09b, Theorem 2.4])

1. *Let $A = (A, \mu)$ be an associative algebra and $\alpha : A \to A$ be a linear map which is multiplicative with respect to μ, i.e. $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$. Then $A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \alpha)$ is a Hom-associative algebra.*

2. *Let $A = (A, [\ ,\ ])$ be a Lie algebra and $\alpha : A \to A$ be a linear map which is multiplicative with respect to $[\ ,\ ]$, i.e. $\alpha \circ [\ ,\ ] = [\ ,\ ] \circ \alpha^{\otimes 2}$. Then $A_\alpha = (A, [\ ,\ ]_\alpha = \alpha \circ [\ ,\ ], \alpha)$ is a Hom-Lie algebra.*

Proof. 1. We have

$$\begin{aligned} \mathfrak{as}_{\mu_\alpha,\alpha} &= \mu_\alpha \circ (\mu_\alpha \otimes \alpha - \alpha \otimes \mu_\alpha) \\ &= \alpha \circ \mu \circ \alpha^{\otimes 2} \circ (\mu \otimes id - id \otimes \mu) \\ &= \alpha^2 \circ \mathfrak{as}_{\mu,id} = 0 \end{aligned}$$

since (A, μ) is associative, so A_α is a Hom-associative algebra.

2. We have $\forall\ x,y \in A,\ [y,x] = -[x,y]$ and

$$\begin{aligned} J_{[\ ,\ ]_\alpha,\alpha} &= [\ ,\ ]_\alpha \circ (\alpha \otimes [\ ,\ ]_\alpha) \circ (id + \sigma + \sigma^2) \\ &= \alpha \circ [\ ,\ ] \circ \alpha^{\otimes 2} \circ (id \otimes [\ ,\ ]) \circ (id + \sigma + \sigma^2) \\ &= \alpha^2 \circ J_{[\ ,\ ],id} = 0 \end{aligned}$$

since $[\ ,\ ]$ is a Lie bracket, so A_α is a Hom-Lie algebra.

□

In particular, if $\alpha : A \to A$ is an (associative or Lie) algebra morphism (A,μ), then it is also a morphism of the Hom-algebra A_α.

Remark 5.1.6 More generally, the categories of Hom-associative algebras and Hom-Lie algebras are closed under twisting by self-weak morphisms. If $A = (A,\mu,\alpha)$ is a Hom-associative algebra (resp. Hom-Lie algebra) and β a weak morphism, then $A_\beta = (A,\mu_\beta = \beta \circ \mu, \beta \circ \alpha)$ is a Hom-associative algebra (resp. Hom-Lie algebra) as well (see [Yau10b]).

For an associative (resp. Lie) algebra (A,μ), if $\alpha : A \to A$ is a morphism, it is a linear map multiplicative for the structure μ. The twisting principle allows to construct a Hom-associative (resp. Hom-Lie) algebra $(A,\mu_\alpha = \alpha \circ \mu, \alpha)$. If the endomorphism α is an automorphism, the notion of Hom-algebra amounts to the data of an algebra and an automorphism of that algebra. If α is not invertible, the process gives new structures.

However, Hom-algebras do not necessarily come from twists of classical structures. The map of Example 5.1.4 of the Hom-Lie Jackson $\mathfrak{sl}(2)$ algebra is not a morphism for the defined bracket but satisfies Hom-Jacobi identity.

5.1.3 Construction of Hom-Lie algebras

We expose various methods to construct Hom-Lie algebras. We use the usual Lie algebra $\mathfrak{sl}(2)$ as a starting point for the different cases, but the constructions works more generally. The Lie algebra $\mathfrak{sl}(2)$ is of dimension 3, generated by elements e,f,h and relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

By twisting principle

To use the previous twisting principle, we must determinate the morphisms of the initial Lie algebra. For $\mathfrak{sl}(2)$ which is simple, an endomorphism $\alpha : \mathfrak{sl}(2) \to \mathfrak{sl}(2)$ has a kernel which is an ideal either reduced to $\{0\}$ or the whole space $\mathfrak{sl}(2)$. In the first case α is an automorphism, in the second case $\alpha \equiv 0$.

Jacobson determinate in [Jac62, Theorem 5,p. 283] the group of automorphisms of $\mathfrak{sl}(2,\mathbb{K})$: it is the set of mappings

$$\begin{aligned} \mathfrak{sl}(2) &\to \mathfrak{sl}(2) \\ X &\mapsto A^{-1}XA \end{aligned} \qquad \text{with} \quad A \in SL(2,\mathbb{K}).$$

The statement is given for a field $\mathbb{K}$ of characteristic 0 algebraically closed, but the result is again true for $\mathbb{K}=\mathbb{R}$.

Example 5.1.7 Considering the matrix representation of $\mathfrak{sl}(2)$ with

$$e=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and a matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$, we obtain Hom-Lie $\mathfrak{sl}(2)_\alpha$ versions of $\mathfrak{sl}(2)$, with α given by

$$\alpha(e)=A^{-1}.e.A=\begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}=d^2e-c^2f+cdh$$

$$\alpha(f)=A^{-1}.f.A=\begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix}=-b^2e+a^2f-abh$$

$$\alpha(h)=A^{-1}.h.A=\begin{pmatrix} bc+ad & 2bd \\ -2ac & -bc-ad \end{pmatrix}=2bde-2acf+(bc+ad)h.$$

Specifying parameters, we find simpler examples of Hom-Lie versions of $\mathfrak{sl}(2)$. Taking $c=b=0$, we have $d=\frac{1}{a}$, and setting $\lambda=d^2$,

$$\begin{aligned} [h,e]_\alpha &= 2\lambda e, & \alpha(e) &= \lambda e, \\ [h,f]_\alpha &= -2\lambda^{-1}f, & \alpha(f) &= \lambda^{-1}f, \\ [e,f]_\alpha &= h, & \alpha(h) &= h; \end{aligned}$$

taking $a=d=0$, we have $c=-\frac{1}{b}$, and setting $\lambda=-c^2$

$$\begin{aligned} [h,e]_\alpha &= 2\lambda f, & \alpha(e) &= \lambda f, \\ [h,f]_\alpha &= -2\lambda^{-1}e, & \alpha(f) &= \lambda^{-1}e, \\ [e,f]_\alpha &= -h, & \alpha(h) &= -h. \end{aligned}$$

With σ-derivations

The original method used in [HLS06] to deform Witt and Virasoro algebras, and which led to the definition of Hom-Lie algebras, uses operators of σ-derivation. This method gives also deformations of the Lie algebra $\mathfrak{sl}(2)$ into Hom-Lie algebras. The following is based on the short presentation of [MS10a].

Let A be a commutative associative $\mathbb{K}$-algebra with unit. We denote by σ an endomorphism of A. A σ-derivation on A is a $\mathbb{K}$-linear map $\partial_\sigma : A \to A$ such that the σ-twisted Leibniz rule holds:

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b). \tag{5.1.6}$$

We let $\mathrm{Der}_\sigma(A)$ denote the A-module of σ-derivations of A. Fixing a homomorphism $\sigma : A \to A$, a σ-derivation $\partial_\sigma \in \mathrm{Der}_\sigma(A)$ and an element $\delta \in A$, we assume that these objects satisfy the two following conditions,

$$\sigma(\mathrm{Ann}(\partial_\sigma)) \subseteq \mathrm{Ann}(\partial_\sigma), \tag{5.1.7}$$

$$\partial_\sigma(\sigma(a)) = \delta\sigma(\partial_\sigma(a)), \qquad \forall\, a \in A, \tag{5.1.8}$$

where $\mathrm{Ann}(\partial_\sigma) = \{a \in A, a \cdot \partial_\sigma = 0\}$. Let $A \cdot \partial_\sigma = \{a \cdot \partial_\sigma, a \in A\}$ denote the cyclic A-submodule of $\mathrm{Der}_\sigma(A)$ generated by ∂_σ and extend σ to $A \cdot \partial_\sigma$ by $\sigma(a \cdot \partial_\sigma) = \sigma(a) \cdot \partial_\sigma$.

The following theorem, from [HLS06], introduces an $\mathbb{K}$-algebra structure on $A \cdot \partial_\sigma$ making it a quasi-Hom-Lie algebra.

Theorem 5.1.8 ([HLS06, Theorem 3 p.11]) *If $\sigma(\mathrm{Ann}(\partial_\sigma)) \subseteq \mathrm{Ann}(\partial_\sigma)$, then the bilinear map $[\ ,\ ]_\sigma$ defined for $a, b \in A$ by*

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma := (\sigma(a) \cdot \partial_\sigma) \circ (b \cdot \partial_\sigma) - (\sigma(b) \cdot \partial_\sigma) \circ (a \cdot \partial_\sigma) \tag{5.1.9}$$

is a well-defined $\mathbb{K}$-algebra product on the $\mathbb{K}$-linear space $A \cdot \partial_\sigma$. It satisfies the following identities for $a, b, c \in A$,

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = (\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)) \cdot \partial_\sigma, \tag{5.1.10}$$

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = -[b \cdot \partial_\sigma, a \cdot \partial_\sigma]_\sigma, \tag{5.1.11}$$

and if, in addition, (5.1.8) *holds, we have the following six-term Jacobi identity*

$$\circlearrowleft_{a,b,c} \Big([\sigma(a) \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma]_\sigma + \delta \cdot [a \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma]_\sigma \Big) = 0. \tag{5.1.12}$$

We apply this on the representation of $\mathfrak{sl}(2)$ in terms of first order differential operators acting on some vector space A of functions in the variable t.

$$e \mapsto \partial, \qquad h \mapsto -2t\partial, \qquad f \mapsto -t^2\partial.$$

Example 5.1.9 Fix $q \in \mathbb{K}^*$. Let $A = \mathbb{K}[t]$ and $\sigma : A \to A$ be the unique endomorphisme defined by $\sigma(t) := qt$, *i.e.* $\sigma(f(t)) = f(qt)$ for $f \in A$, and $\partial_\sigma : A \to A$ the σ-derivation defined by $\partial_\sigma(t) = t$, *i.e.* $\partial_\sigma(t^n) = \{n\}_q t^n$, with q-analogues $\{n\}_q = 1 + q + \cdots + q^{n-1}$.

Set S the $\mathbb{K}$-linear subspace of $A \cdot \partial_\sigma$ spanned by elements $E := \partial_\sigma$, $H := -2t\partial_\sigma$, $F := -t^2\partial_\sigma$. The element $\delta = q$ satisfies equation

(5.1.8), so with the previous bracket $[\ ,\ ]_\sigma$ and the $\mathbb{K}$-linear map $\alpha = \sigma + q\,id$, $(S, [\ ,\ ]_\sigma, \alpha)$ is a Hom-Lie algebra. We have

$$[H,E]_\sigma = 2E, \qquad \alpha(E) = (1+q)E,$$
$$[H,F]_\sigma = -2qF, \qquad \alpha(F) = q(1+q)F,$$
$$[E,F]_\sigma = \frac{q+1}{2}H, \quad \alpha(H) = 2qH.$$

For $q = 1$, brackets are those of $\mathfrak{sl}(2)$ but $\alpha = 2id$.

Example 5.1.10 Slightly adapting the previous relations, we obtain a q-deformation family of Witt algebras. We consider this time $A = \mathbb{C}[t, t^{-1}]$, and, with the previous notations, $D := t\partial_\sigma = -\dfrac{id - \sigma}{q-1}$. Then

$$\mathcal{D}_q := \mathrm{Der}_\sigma(A) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot d_n \qquad \text{where} \quad d_n = -t^n \cdot D.$$

With the bracket $[\ ,\ ] : \mathcal{D}_q \to \mathcal{D}_q$ defined on generators by $[d_n, d_m] := \left(\{n\}_q - \{m\}_q\right) d_{n+m}$ and the linear map $\alpha = \sigma + id$, defined on elements by $\alpha(d_n) = (q^n + 1)d_n$, $(\mathcal{D}_q, [\ ,\ ], \alpha)$ is a Hom-Lie algebra.

For $q = 1$, we recover the classical Witt algebra.

Hom-Lie structure associated with a bracket

An other way to construct Hom-Lie algebras is to compute the various possible linear maps (not necessarily multiplicatives) for a given bracket. We apply this idea to the previous exemple..

Example 5.1.11 Considering brackets $[h,e] = 2e,\ [h,f] = -2qf,\ [e,f] = \dfrac{q+1}{2}h$, we compute maps α written as $\alpha = (\alpha_{ij})$ satisfying Hom-Jacobi identity. We solve equations $\circlearrowleft_{x,y,z}[\alpha(x),[y,z]] = 0$ on the basis, in the coefficients α_{ij}. We obtain Hom-Lie algebras with the previous brackets and map α such that

$$\alpha(e) = ae + bf + ch,$$
$$\alpha(f) = de + aqf + k\frac{1+q}{4}h,$$
$$\alpha(h) = ke + c\frac{4q}{1+q}f + lh,$$

with parameters $a, b, c, d, k, l \in \mathbb{K}$. Details of the computation are given in annex Section A.2.

Setting $b = c = d = k = 0$, $a = 1 + q$, $l = 2q$, we obtain Example 5.1.9, and for $b = c = d = k = 0$, $a = l = q$, we get Example 5.1.4. These maps are never multiplicative for the brackets of the usual Lie algebra $\mathfrak{sl}(2)$, so these Hom-Lie algebras do not come from the classical Lie algebra $\mathfrak{sl}(2)$ by twisting principle Theorem 5.1.5. Also, these maps are not multiplicatives for the considered bracket.

In particular, for $q = 1$, Hom-Lie algebras with the same bracket that $\mathfrak{sl}(2)$ are obtained with the following linear maps α (mentioned in [MS10b]),

$$\alpha(e) = ae + bf + ch,$$
$$\alpha(f) = de + af + \frac{1}{2}kh,$$
$$\alpha(h) = ke + 2cf + lh.$$

5.2 Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras

In this section we summarize and describe some of basic properties of Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras which generalize the classical coalgebra, bialgebra and Hopf algebra structures.

5.2.1 Hom-coalgebras and duality

Definition 5.2.1 A *Hom-coalgebra* is a triple (A, Δ, α) where A is a $\mathbb{K}$-module and $\Delta : A \to A \otimes A$, $\alpha : A \to A$ are linear maps.

A *Hom-coassociative coalgebra* is a Hom-coalgebra satisfying

$$(\alpha \otimes \Delta) \circ \Delta = (\Delta \otimes \alpha) \circ \Delta. \tag{5.2.1}$$

A Hom-coassociative coalgebra is said to be *counital* if there exists a map $\varepsilon : A \to \mathbb{K}$ satisfying

$$(id \otimes \varepsilon) \circ \Delta = \alpha \qquad \text{and} \qquad (\varepsilon \otimes id) \circ \Delta = \alpha. \tag{5.2.2}$$

Let (A, Δ, α) and (A', Δ', α') be two Hom-coalgebras (resp. Hom-coassociative coalgebras). A linear map $f : A \to A'$ is a *morphism of Hom-coalgebras* (resp. *Hom-coassociative coalgebras*) if

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

If furthermore the Hom-coassociative coalgebras admit counits ε and ε', we have moreover $\varepsilon = \varepsilon' \circ f$.

The following theorem shows how to construct a new Hom-coassociative Hom-coalgebra starting with a Hom-coassociative Hom-coalgebra and a Hom-coalgebra morphism. We only need the coassociative comultiplication of the coalgebra.

Theorem 5.2.2 *Let $(A, \Delta, \alpha, \varepsilon)$ be a counital Hom-coalgebra and $\beta : A \to A$ be a weak Hom-coalgebra morphism. Then $(A, \Delta_\beta, \alpha \circ \beta, \varepsilon)$, where $\Delta_\beta = \Delta \circ \beta$, is a counital Hom-coassociative coalgebra.*

Proof. We show that $(A, \Delta_\beta, \alpha \circ \beta)$ satisfies the axiom (5.2.1).

Indeed, using the fact that $(\beta \otimes \beta) \circ \Delta = \Delta \circ \beta$, we have

$$\begin{aligned}\left(\alpha \circ \beta \otimes \Delta_\beta\right) \circ \Delta_\beta &= (\alpha \circ \beta \otimes \Delta \circ \beta) \circ \Delta \circ \beta \\ &= ((\alpha \otimes \Delta) \circ \Delta) \circ \beta^2 \\ &= ((\Delta \otimes \alpha) \circ \Delta) \circ \beta^2 \\ &= \left(\Delta_\beta \otimes \alpha \circ \beta\right) \circ \Delta_\beta.\end{aligned}$$

Moreover, the axiom (5.2.2) is also satisfied, since we have

$$(id \otimes \varepsilon) \circ \Delta_\beta = (id \otimes \varepsilon) \circ \Delta \circ \beta = \alpha \circ \beta = (\varepsilon \otimes id) \circ \Delta \circ \beta = (\varepsilon \otimes id) \circ \Delta_\beta.$$

□

Remark 5.2.3 The previous theorem shows that the category of coassociative Hom-coalgebras is closed under weak Hom-coalgebra morphisms. It leads to the following examples:

1. Let (A, Δ) be a coassociative coalgebra and $\beta : A \to A$ be a coalgebra morphism. Then (A, Δ_β, β) is a Hom-coassociative coalgebra.
2. Let (A, Δ, α) be a multiplicative coassociative Hom-coalgebra. For all non negative integer n, $(A, \Delta_{\alpha^n}, \alpha^{n+1})$ is a Hom-coassociative coalgebra.

In the following we show that there is a duality between Hom-associative and Hom-coassociative structures (see [SPAS09, MS10a]).

Theorem 5.2.4 *Let (A, Δ, α) be a coalgebra Hom-coassociative. Then the dual $(A^*, \Delta^*, \alpha^*)$ is a Hom-associative algebra. Moreover A^* is unital if A is counital.*

Proof. The product $\mu = \Delta^*$ is defined from $A^* \otimes A^*$ to A^* by

$$(fg)(x) = \Delta^*(f, g)(x) = \langle \Delta(x), f \otimes g \rangle = (f \otimes g)(\Delta(x)) = \sum_{(x)} f(x_1) g(x_2), \quad \forall x \in A$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between the vector space $A \otimes A$ and its dual vector space. For $f, g, h \in A^*$ and $x \in A$, we have

$$(fg)\alpha^*(h)(x) = \langle (\Delta \otimes \alpha) \circ \Delta(x), f \otimes g \otimes h \rangle$$

and

$$\alpha^*(f)(gh)(x) = \langle (\alpha \otimes \Delta) \circ \Delta(x), f \otimes g \otimes h \rangle$$

so the Hom-associativity $\mu \circ (\mu \otimes \alpha^* - \alpha^* \otimes \mu) = 0$ comes from the Hom-coassociativity $(\Delta \otimes \alpha - \alpha \otimes \Delta) \circ \Delta = 0$.
Moreover, if A has a counit ε satisfying $(id \otimes \varepsilon) \circ \Delta = \alpha = (\varepsilon \otimes id) \circ \Delta$ then for $f \in A^*$ and $x \in A$ we have

$$(\varepsilon f)(x) = \sum_{(x)} \varepsilon(x_1) f(x_2) = \sum_{(x)} f(\varepsilon(x_1) x_2) = f(\alpha(x)) = \alpha^*(f)(x)$$

and

$$(f\varepsilon)(x) = \sum_{(x)} f(x_1)\varepsilon(x_2) = \sum_{(x)} f(x_1 \varepsilon(x_2)) = f(\alpha(x)) = \alpha^*(f)(x)$$

which shows that ε is the unit in A^*. □

The dual of a Hom-algebra (A, μ, α) is not always a Hom-coalgebra, because the coproduct does not land in the good space: $\mu^* : A^* \to (A \otimes A)^* \supsetneq A^* \otimes A^*$. It is the case if the Hom-algebra is of finite dimension, since $(A \otimes A)^* = A^* \otimes A^*$.

In the general case, for any algebra A, define

$$A^\circ = \{f \in A^*, f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\},$$

where a *cofinite ideal* I is an ideal $I \subset A$ such that A/I is finite dimensional.

A° is a subspace of A^* since it is closed under multiplication by scalars and the sum of two elements of A° is again in A° since the intersection of two cofinite ideals is again such. If A is finite dimensional, of course $A^\circ = A^*$.

Lemma 5.2.5 *Let (A, μ_A, α_A) and (B, μ_B, α_B) be two Hom-associative algebras and $f : A \to B$ a morphism of Hom-algebras. Then the dual map $f^* : B^* \to A^*$ satisfies $f^*(B^\circ) \subset A^\circ$.*

Proof. Let J be a cofinite ideal of B and $p : B \to B/J$ the canonical projection. We set $\widetilde{f} = p \circ f : A \to B/J$.

Remark that $f^{-1}(J)$ is an ideal of A. Indeed, for $x \in A$ we have $f(xf^{-1}(J)) = f(x)f(f^{-1}(J)) = f(x)J \subset J$. So $xf^{-1}(J) \subset f^{-1}(J)$. Moreover $\alpha_A(f^{-1}(J)) = f^{-1}(\alpha_B(J)) \subset f^{-1}(J)$.

The sequence

$$0 \to f^{-1}(J) \xrightarrow{i} A \xrightarrow{\widetilde{f}} B/J \to 0$$

is exact. Define a map $f_\star : A/f^{-1}(J) \to B/J$ by $f_\star(x + f^{-1}(J)) = f(x)$. It induces an isomorphism $A/f^{-1}(J) \to \operatorname{Im} \widetilde{f}$. Thus $A/f^{-1}(J)$ is finite-dimensional.

Similarly, we have $f^*(B^\circ) \subset A^\circ$. Indeed, let $b^* \in B^*$ such that $\operatorname{Ker}(b^*) \supset J$. Then $\operatorname{Ker}(f^*(b^*)) \supset f^{-1}(J)$, because

$$\langle f^*(b^*), f^{-1}(J) \rangle = \langle b^*, f(f^{-1}(J)) \rangle = \langle b^*, J \rangle = 0.$$

□

Using this lemma, we can show in a similar way as [Swe69, Lemme 6.0.1] that $A^\circ \otimes A^\circ = (A \otimes A)^\circ$ and the dual $\mu^* : A^* \to (A \otimes A)^*$ of the multiplication $\mu : A \otimes A \to A$ satisfies $\mu^*(A^\circ) \subset A^\circ \otimes A^\circ$. Indeed, for $f \in A^*$, $x, y \in A$, we have $\langle \mu^*(f), x \otimes y\rangle = \langle f, xy\rangle$. Hence, if I is a cofinite ideal such that $f(I) = 0$, then $I \otimes A + A \otimes I$ is a cofinite ideal of $A \otimes A$ on which $\mu^*(f)$ vanishes.

For a map $f : A \to B$ we note $f^\circ := f^*|_{B^\circ} : B^\circ \to A^\circ$. Set $\Delta := \mu^\circ = \mu^*|_{A^\circ}$ and $\varepsilon : A^\circ \to \mathbb{K}$ defined by $\varepsilon(f) = f(1)$.

Theorem 5.2.6 *Let (A, μ, α) be a multiplicative Hom-associative algebra. Then $(A^\circ, \Delta, \alpha^\circ)$ is an Hom-coalgebra. Moreover, it is counital if A is unital.*

Proof. The coproduct Δ is defined from A° to $A^\circ \otimes A^\circ$ by

$$\Delta(f)(x \otimes y) = \mu^*|_{A^\circ}(f)(x \otimes y) = \langle \mu(x \otimes y), f\rangle = f(xy), \quad x, y \in A.$$

For $f, g, h \in A^\circ$ and $x, y \in A$, we have

$$(\Delta \circ \alpha^\circ) \circ \Delta(f)(x \otimes y \otimes z) = \langle \mu \circ (\mu \otimes \alpha)(x \otimes y \otimes z), f\rangle$$

and

$$(\alpha^\circ \circ \Delta) \circ \Delta(f)(x \otimes y \otimes z) = \langle \mu \circ (\otimes \mu \alpha)(x \otimes y \otimes z), f\rangle$$

so the Hom-coassociativity $(\Delta \otimes \alpha^\circ - \alpha^\circ \otimes \Delta) \circ \Delta = 0$ comes from the Hom-associativity $\mu \circ (\mu \otimes \alpha - \alpha \otimes \mu) = 0$.
Moreover, if A has a unit η satisfying $\mu \circ (id \otimes \eta) = \alpha = \mu \circ (\eta \otimes id)$ then for $f \in A^\circ$ and $x \in A$ we have

$$(\varepsilon \otimes id) \circ \Delta(f)(x) = f(1.x) = f(\alpha(x)) = \alpha^\circ(f)(x)$$

and

$$(id \otimes \varepsilon) \circ \Delta(f)(x) = f(x.1) = f(\alpha(x)) = \alpha^\circ(f)(x)$$

which shows that $\varepsilon : A^\circ \to \mathbb{K}$, $f \mapsto f(1)$ is the counit in A°. □

If the Hom-associative algebra is finite-dimensional, its dual is a Hom-coassociative coalgebra.

5.2.2 Hom-bialgebra and Hom-Hopf algebra

Definition 5.2.7 A *Hom-bialgebra* is a tuple $(A, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ where

1. (A, μ, α, η) is a Hom-associative algebra with a unit η.

2. $(A, \Delta, \beta, \varepsilon)$ is a Hom-coassociative coalgebra with a counit ε.

3. The linear maps Δ and ε are compatible with the multiplication μ and the unit η, that is

$$\Delta(e) = e \otimes e \qquad \text{where } e = \eta(1),$$
$$\Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x_1 \otimes y_1) \otimes \mu(x_2 \otimes y_2),$$
$$\varepsilon(e) = 1,$$
$$\varepsilon(\mu(x \otimes y)) = \varepsilon(x)\,\varepsilon(y),$$
$$\varepsilon \circ \alpha(x) = \varepsilon(x),$$

where the bullet $\bullet$ denotes the multiplication on tensor product.

If $\alpha = \beta$ the Hom-bialgebra is noted $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$.

Combining previous observations, with only one twisting map, we obtain:

Proposition 5.2.8 *Let (A, μ, Δ, α) be a Hom-bialgebra. Then the finite dual $(A^\circ, \mu^\circ, \Delta^\circ, \alpha^\circ)$ is a Hom-bialgebra as well.*

Remark 5.2.9

1. Given a Hom-bialgebra $(A, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, it is shown in [SPAS09, MS10a] that the vector space $\mathrm{Hom}(A, A)$ with the multiplication given by the convolution product carries a structure of Hom-associative algebra.
2. An endomorphism S of A is said to be an *antipode* if it is the inverse of the identity over A for the Hom-associative algebra $\mathrm{Hom}(A, A)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta$$

and the unit being $\eta \circ \varepsilon$.

3. A *Hom-Hopf algebra* is a Hom-bialgebra with an antipode.

By combining Theorem 5.1.5 and Theorem 5.2.2, we obtain:

Proposition 5.2.10 *Let (A, μ, Δ, α) be a Hom-bialgebra and $\beta : A \to A$ be a Hom-bialgebra morphism commuting with α. Then $(A, \mu_\beta, \Delta_\beta, \beta\circ\alpha)$ is a Hom-bialgebra.*

Notice that the category of Hom-bialgebra is not closed under weak Hom-bialgebra morphisms.

Example 5.2.11 (Universal enveloping Hom-algebra) Given a Hom-associative algebra $A = (A, \mu, \alpha)$, one can associate to it a Hom-Lie algebra $HLie(A) = (A, [\ ,\ ], \alpha)$ with the same underlying module (A, α) and the bracket

$[\ ,\ ] = \mu - \mu^{op}$. This construction gives a functor $HLie$ from Hom-associative algebras to Hom-Lie algebras. In [Yau08], Yau constructed the left adjoint U_{HLie} of $HLie$. He also made some minor modifications in [Yau10a] to take into account the unital case.

The functor U_{HLie} is defined as

$$U_{HLie}(A) = F_{HNAs}(A)/I^{\infty} \qquad \text{with} \quad F_{HNAs}(A) = \bigoplus_{n\in\mathbb{N}} \bigoplus_{\tau\in T_n^{wt}} A_{\tau}^{\otimes n} \tag{5.2.3}$$

for a Hom-Lie algebra $(A,[\ ,\ ],\alpha)$. Here T_n^{wt} is the set of weighted n-trees encoding the multiplication of elements (by trees) and twisting by α (by weights), $A_{\tau}^{\otimes n}$ is a copy of $A^{\otimes n}$ and I^{∞} is a certain submodule of relations build in such a way that the quotient is Hom-associative.

Moreover, the comultiplication $\Delta : U_{HLie}(A) \to U_{HLie}(A) \otimes U_{HLie}(A)$ defined on A by $\Delta(x) = \alpha(x)\otimes\mathbf{1} + \mathbf{1}\otimes\alpha(x)$ equips the Hom-associative algebra $U_{HLie}(A)$ with a structure of Hom-bialgebra.

5.3 Hom-Lie coalgebras and Hom-Lie bialgebras

As it is the case for the Hom-associative algebras, the Hom-Lie algebras also have a dualized version, Hom-Lie coalgebras. They share the same kind of properties. We review here the principal results, similar results can be found in [Yau09a].

Definition 5.3.1 A *Hom-Lie coalgebra* (A,Δ,α) is a Hom-coalgebra satisfying $\Delta + \Delta^{op} = 0$ and the Hom-coJacobi identity

$$(id + \sigma + \sigma^2) \circ (\alpha \otimes \Delta) \circ \Delta = 0. \tag{5.3.1}$$

We call Δ the cobracket.

We recover a Lie coalgebra when $\alpha = id$. Just like (co)associative (co)algebras we have the following dualization properties.

Theorem 5.3.2

1. *Let (A,Δ,α) be a Hom-Lie coalgebra. Then (A^*,Δ^*,α^*) is an Hom-Lie algebra.*

2. *Let $(A,[\ ,\ ],\alpha)$ be a Hom-Lie algebra. Then $(A^{\circ},[\ ,\ ]^{\circ},\alpha^{\circ})$ is an Hom-Lie coalgebra.*

The twisting principle also works, showing that the category of Hom-Lie coalgebras is closed under weak Hom-coalgebras morphisms.

Theorem 5.3.3 *Let (A,Δ,α) be a Hom-Lie coalgebra and $\beta : A \to A$ a weak Hom-coalgebra morphism. Then $(A,\Delta_\beta = \Delta\circ\beta, \alpha\circ\beta)$ is a Hom-Lie coalgebra.*

Proof. We have $\Delta_\beta + \Delta_\beta^{op} = (\Delta+\Delta^{op})\circ\beta = 0$, and

$$\begin{aligned}(id+\sigma+\sigma^2)\circ(\alpha\circ\beta\otimes\Delta_\beta)\circ\Delta_\beta &= (id+\sigma+\sigma^2)\circ(\alpha\circ\beta\otimes\Delta\circ\beta)\circ\Delta\circ\beta\\ &= (id+\sigma+\sigma^2)\circ(\alpha\otimes\Delta)\circ\Delta\circ\beta^2\\ &= 0.\end{aligned}$$

□

The previous theorem can be used to construct Hom-Lie coalgebras.

Corollary 5.3.4

1. *Let (A,Δ) be a Lie coalgebra and $\beta : A \to A$ be a Lie coalgebra morphism. Then (A,Δ_β,β) is a Hom-Lie coalgebra.*
2. *Let (A,Δ,α) be a multiplicative Hom-Lie coalgebra. For all non negative integer n, $(A,\Delta_{\alpha^n},\alpha^{n+1})$ is a Hom-Lie coalgebra.*

The Hom-Lie bialgebra structure was introduced first in [Yau09a]. The definition presented below is slightly more general. They border the class defined by Yau and permit to consider the compatibility condition for different A-valued cohomology of Hom-Lie algebras.

Definition 5.3.5 A *Hom-Lie bialgebra* is a tuple $(A,[\ ,\ ],\alpha,\Delta,\beta)$ where

1. $(A,[\ ,\ ],\alpha)$ is a Hom-Lie algebra.
2. (A,Δ,β) is a Hom-Lie coalgebra.
3. The following compatibility condition holds for $x,y\in A$:

$$\Delta([x,y]) = ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x)), \tag{5.3.2}$$

where the adjoint map $ad_x : A^{\otimes n}\to A^{\otimes n}$ ($n\geqslant 2$) is given by

$$ad_x(y_1\otimes\cdots\otimes y_n) = \sum_{i=1}^{n}\beta(y_1)\otimes\cdots\otimes\beta(y_{i-1})\otimes[x,y_i]\otimes\beta(y_{i+1})\otimes\cdots\otimes\beta(y_n). \tag{5.3.3}$$

A *morphism* $f : A\to A'$ of Hom-Lie bialgebras is a linear map commuting with α and β such that $f\circ[\ ,\ ] = [\ ,\ ]\circ f^{\otimes 2}$ and $\Delta\circ f = f^{\otimes 2}\circ\Delta$.

If $\alpha=\beta=id$ we recover Lie bialgebras and if $\alpha=\beta$, we recover the class defined in [Yau09a], these Hom-Lie bialgebras are denoted $(A,[\ ,\ ],\Delta,\alpha)$.

In terms of elements, the compatibility condition (5.3.2) writes

$$\begin{aligned}\Delta([x,y]) =& ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x))\\ =& [\alpha(x),y_1]\otimes\beta(y_2)+\beta(y_1)\otimes[\alpha(x),y_2]\\ & -[\alpha(y),x_1]\otimes\beta(x_2)-\beta(x_1)\otimes[\alpha(y),x_2].\end{aligned}\tag{5.3.4}$$

Remark 5.3.6 If $\alpha=\beta$, the compatibility condition (5.3.4) is equivalent to say that Δ is a 1-cocycle with respect to α^0-adjoint cohomology of Hom-Lie algebras and for $\beta = id$, it corresponds to α^{-1}-adjoint cohomology, (see [She11] and [AEM11]).

The following Proposition generalizes slightly [Yau09a, Theorem 3.5].

Proposition 5.3.7 *Let $(A,[\ ,\ ],\Delta,\alpha)$ be a Hom-Lie bialgebra and $\beta : A \to A$ a Hom-Lie bialgebra morphism commuting with α. Then $(A,[\ ,\ ]_\beta=\beta\circ[\ ,\ ],\Delta_\beta = \Delta\circ\beta,\alpha\circ\beta)$ is a Hom-Lie bialgebra.*

Proof. We already know that $(A,[\ ,\ ]_\beta,\beta\circ\alpha)$ is a Hom-Lie algebra and that $(A,\Delta_\beta,\alpha\circ\beta)$ is a Hom-Lie coalgebra. It remains to prove the compatibility condition (5.3.4) for Δ_β and $[\ ,\ ]_\beta$, with the twisting map $\alpha\circ\beta=\beta\circ\alpha$. On one side, we have

$$\Delta_\beta([x,y]) = \Delta\circ\beta^2\circ[x,y] = \left(\beta^{\otimes 2}\right)^2\circ\Delta([x,y]),$$

since $\Delta\circ\beta=\beta^{\otimes 2}\circ\Delta$. Using in addition $\beta\circ[\ ,\ ]=[\ ,\ ]\circ\beta^{\otimes 2}$ and the fact that α and β commute, we have

$$\begin{aligned}ad_{\alpha\circ\beta(x)}(\Delta_\beta(y)) &= ad_{\alpha\circ\beta(x)}(\beta(y_1)\otimes\beta(y_2))\\ &= [\alpha\circ\beta(x),\beta(y_1)]_\beta\otimes\alpha\circ\beta^2(y_2)+\alpha\circ\beta^2(y_1)\otimes[\alpha\circ\beta(x),\beta(y_2)]_\beta\\ &= \left(\beta^{\otimes 2}\right)^2([\alpha(x),y_1]\otimes\alpha(y_2)+\alpha(y_1)\otimes[\alpha(x),y_2]).\end{aligned}$$

It follows that $\Delta_\beta([x,y]) = ad_{\alpha\circ\beta(x)}(\Delta_\beta(y)) - ad_{\alpha\circ\beta(y)}(\Delta_\beta(x))$ as wished. □

As for the Hom-bialgebra, the category of Hom-Lie bialgebra is not closed under weak Hom-Lie bialgebra morphisms.

Hom-Lie bialgebra can be dualized. We obtain the following proposition which generalizes the result stated in [Yau09a] for finite dimensional case using natural pairing.

Proposition 5.3.8 *Let $(A,[\ ,\ ],\alpha,\Delta,\beta)$ be a multiplicative Hom-Lie bialgebra. Then the finite dual $(A^\circ,[\ ,\ ]^\circ,\alpha^\circ,\Delta^\circ,\beta^\circ)$ is a Hom-Lie bialgebra as well.*

6

Hom-(co)Poisson structures

Contents

Hom-Poisson algebras also satisfy the twisting principle, such a morphism is computed for the Sklyanin algebra. We show the links between Hom-Poisson algebras and 1-operation structures, then study their duals, which are linked to Hom-Lie bialgebras.

6.1 Hom-Poisson algebras

We give here the definition of Hom-Poisson algebras and show that the twisting principle also holds in that case. We present in finite dimension a way to build Hom-Poisson algebras from Hom-Lie algebras.

6.1.1 Definitions and examples

Definition 6.1.1 A *Hom-Poisson* algebra is tuple $(A,\mu,\{\ ,\ \},\alpha)$ consisting of

(1) a commutative Hom-associative algebra (A,μ,α) and

(2) a Hom-Lie algebra $(A,\{\ ,\ \},\alpha)$

such that the Hom-Leibniz identity

$$\{\ ,\ \}\circ(\mu\otimes\alpha)=\mu\circ(\alpha\otimes\{\ ,\ \}+(\{\ ,\ \}\otimes\alpha)\circ\tau_{23}) \tag{6.1.1}$$

is satisfied.

In a Hom-Poisson algebra $(A,\{\ ,\ \},\mu,\alpha)$, the operation $\{\ ,\ \}$ is called *Hom-Poisson bracket*. In terms of elements $x,y,z\in A$, the Hom-Leibniz identity says

$$\{xy,\alpha(z)\}=\alpha(x)\{y,z\}+\{x,z\}\alpha(y)$$

where as usual $\mu(x,y)$ is abbreviated to xy. By the antisymmetry of the Hom-Poisson bracket $\{\ ,\ \}$, the Hom-Leibniz identity is equivalent to

$$\{\alpha(x),yz\}=\{x,y\}\alpha(z)+\alpha(y)\{x,z\}.$$

We recover Poisson algebras when the twisting map is the identity.

Definition 6.1.2 A *Hom-Poisson bialgebra* $(A,\mu,\eta,\Delta,\varepsilon,\alpha,\{\ ,\ \})$ is a Hom-Poisson algebra $(A,\mu,\{\ ,\ \},\alpha)$ which is also a Hom-bialgebra $(A,\mu,\eta,\Delta,\varepsilon,\alpha)$, the two structures being compatible in the sense that $\{\ ,\ \}$ is a coderivation along μ,

$$\Delta\circ\{\ ,\ \}=(\{\ ,\ \}\otimes\mu+\mu\otimes\{\ ,\ \})\circ\Delta^{[2]}.$$

In term of elements, this compatibility condition writes

$$\Delta(\{a,b\})=\{\Delta(a),\Delta(b)\}$$

with the Hom-Poisson bracket on $A\otimes A$ defined by

$$\{a_1\otimes a_2,b_1\otimes b_2\}:=\{a_1,b_1\}\otimes a_2b_2+a_1b_1\otimes\{a_2,b_2\}.$$

We have the same definition for Hom-Poisson Hopf algebras.

Example 6.1.3 Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over $\mathbb{K}$. The following multiplication μ, skew-symmetric bracket $\{\ ,\ \}$ and linear map α on A define a Hom-Poisson algebra over $\mathbb{K}^3$:

$$\mu(x_1, x_1) = x_1, \qquad \{x_1, x_2\} = ax_2 + bx_3,$$
$$\mu(x_1, x_2) = \mu(x_2, x_1) = x_3, \qquad \{x_1, x_3\} = cx_2 + dx_3,$$

$$\alpha(x_1) = \lambda_1 x_2 + \lambda_2 x_3, \quad \alpha(x_2) = \lambda_3 x_2 + \lambda_4 x_3, \quad \alpha(x_3) = \lambda_5 x_2 + \lambda_6 x_3$$

where $a, b, c, d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ are parameters in $\mathbb{K}$.

6.1.2 Twisting principle

Theorem 6.1.4 ([Yau10b, Theorem 3.2]) *Let $A = (A, \mu, \{\ ,\ \})$ be a Poisson algebra, and $\alpha : A \to A$ be a linear map which is multiplicative for μ and $\{\ ,\ \}$. Then $A_\alpha = (A, \quad \mu_\alpha = \alpha \circ \mu, \quad \{\ ,\ \}_\alpha = \alpha \circ \{\ ,\ \}, \quad \alpha)$ is a Hom-Poisson algebra.*

Proof. Using Theorem 5.1.5, we already have that (A, μ_α, α) is a commutative Hom-associative algebra and that $(A, \{\ ,\ \}_\alpha, \alpha)$ is a Hom-Lie algebra. It remains to check the Hom-Leibniz identity

$$\begin{aligned}\{\ ,\ \}_\alpha \circ (\mu_\alpha \otimes \alpha) &= \alpha \circ \{\ ,\ \} \circ \alpha^{\otimes 2} \circ (\mu \otimes id)\\ &= \alpha^2 \circ \{\ ,\ \} \circ (\mu \otimes id),\end{aligned}$$

since A is a Poisson algebra

$$\begin{aligned}&= \alpha^2 \circ \mu \circ (id \otimes \{\ ,\ \} + (\{\ ,\ \} \otimes id) \circ \tau_{23})\\ &= \alpha \circ \mu \circ \alpha^{\otimes 2} \circ (id \otimes \{\ ,\ \} + (\{\ ,\ \} \otimes id) \circ \tau_{23})\\ &= \mu_\alpha \circ (\alpha \otimes \{\ ,\ \}_\alpha + (\{\ ,\ \}_\alpha \otimes \alpha) \circ \tau_{23}),\end{aligned}$$

so A_α is a Hom-Poisson algebra. □

6.1.3 Application to Sklyanin algebra

Example 6.1.5 The Sklyanin Poisson algebra $q_4(\mathcal{E})$ (see [ORTP10] for a more detailed definition and properties) is defined on $\mathbb{C}[x_0, x_1, x_2, x_3]$ by a parameter $k \in \mathbb{C}$ with the usual polynomial multiplication, and bracket given by $\{\ ,\ \}$ with brackets between the coordinate functions defined as

$$\begin{aligned}\{x_i, x_{i+1}\} &= k^2 x_i x_{i+1} - x_{i+2} x_{i+3},\\ \{x_i, x_{i+2}\} &= k(x_{i+3}^2 - x_{i+1}^2),\end{aligned} \qquad i = 0, 1, 2, 3 \pmod 4.$$

We again search a morphism $\alpha : q_4(\mathcal{E}) \to q_4(\mathcal{E})$ written as $\alpha = (\alpha_{ij})$ with respect to the basis (x_0, x_1, x_2, x_3), by solving the coefficients α_{ij} in the equations $\alpha([x_i, x_j]) = [\alpha(x_i), \alpha(x_j)]$ with respect to the basis. For simplicity, we take α diagonal, $\alpha_{ij} = 0$ if $i \neq j$.
We obtain $q_4(\mathcal{E})_\alpha$, Hom-Poisson versions of $q_4(\mathcal{E})$, with α given by

1. $\alpha(x_0) = -\lambda x_0,\ \alpha(x_1) = i\lambda x_1,\ \alpha(x_2) = \lambda x_2,\ \alpha(x_3) = -i\lambda x_3,$
2. $\alpha(x_0) = -\lambda x_0,\ \alpha(x_1) = -i\lambda x_1,\ \alpha(x_2) = \lambda x_2,\ \alpha(x_3) = i\lambda x_3,$
3. $\alpha(x_0) = \lambda x_0,\ \alpha(x_1) = -\lambda x_1,\ \alpha(x_2) = \lambda x_2,\ \alpha(x_3) = -\lambda x_3,$
4. $\alpha = \lambda id,$

with $\lambda \in \mathbb{K}$.

For example, $q_4(\mathcal{E})$ carries a structure of Hom-Poisson algebra, for any $\lambda \in \mathbb{C}$, with the following bracket

$$\begin{aligned} &\{x_i, x_{i+1}\} = -\lambda^2(k^2 x_i x_{i+1} - x_{i+2}x_{i+3}), \\ &\{x_i, x_{i+2}\} = \lambda^2 k(x_{i+3}^2 - x_{i+1}^2), \end{aligned} \qquad i = 1,2,3,4 \pmod 4,$$

and linear map

$$\alpha(x_0) = \lambda x_0,\ \alpha(x_1) = -\lambda x_1,\ \alpha(x_2) = \lambda x_2,\ \alpha(x_3) = -\lambda x_3.$$

6.1.4 Constructing Hom-Poisson algebras from Hom-Lie algebras

Suppose that $(A, [\ ,\ ], \alpha)$ is a finite-dimensional Hom-Lie algebra and $\{e_i\}_{1\leqslant i\leqslant n}$ is a basis of A. Let C_{ij}^k be the structure constants of the bracket with respect to the basis, that is $[e_i, e_j] = \sum_{k=1}^n C_{ij}^k e_k$ and α_i^s be the coefficients of the morphism α, that is $\alpha(e_i) = \sum_{s=1}^n \alpha_i^s e_s$. The skew-symmetry of the bracket and the Hom-Jacobi condition can be written with the structure constants as

$$C_{ji}^k = -C_{ij}^k \qquad \text{antisymmetry,}$$

$$\sum_{1\leqslant p,q\leqslant n} (\alpha_i^p C_{jk}^q + \alpha_j^p C_{ki}^q + \alpha_k^p C_{ij}^q) C_{pq}^s = 0 \qquad \text{Hom-Jacobi identity.}$$

To construct a Hom-Poisson algebra from a Hom-Lie algebra, we should define a commutative multiplication $\cdot$ which is Hom-associative and a bracket $\{\ ,\ \}$ satisfying the Hom-Leibniz identity. Define the bracket $\{\ ,\ \}$ as being equal to the bracket $[\ ,\ ]$ on the basis, and extended by the Hom-Leibniz identity.

Let M_{ij}^k be the structure constants for the multiplication, that is $e_i \cdot e_j = \sum_{k=1}^n M_{ij}^k e_k$. By commutativity, $M_{ji}^k = M_{ij}^k$. The Hom-Leibniz identity writes

$$\begin{aligned} & 0 = \{e_i \cdot e_j, \alpha(e_k)\} - \alpha(e_i)\cdot\{e_j, e_k\} - \{e_i, e_k\}\cdot\alpha(e_j) \\ \Leftrightarrow\ & 0 = \sum_{s=1}^n \underbrace{\left(M_{ij}^p \alpha_k^q C_{pq}^s - (\alpha_i^p C_{jk}^q + C_{ik}^p \alpha_j^q) M_{pq}^s\right)}_{S_{ijks}} e_s \\ \Leftrightarrow\ & 0 = S_{ijks}, \end{aligned}$$

giving a linear system in the M_{ij}^l of n^4 equations in n^3 unknowns[1].

The Hom-associativity writes

$$\begin{aligned} 0 &= (e_i \cdot e_j) \cdot \alpha(e_k) - \alpha(e_i) \cdot (e_j \cdot e_k) \\ \Leftrightarrow 0 &= \sum_{s=1}^{n} \underbrace{\left((M_{ij}^p \alpha_k^q - \alpha_i^p M_{jk}^q) M_{pq}^s\right)}_{R_{ijks}} e_s \\ \Leftrightarrow 0 &= R_{ijks}, \end{aligned}$$

giving a non linear system in the M_{ij}^l of n^4 equations in n^3 unknowns.

Solving first the equations of Hom-Leibniz and then checking if the solutions satisfy the Hom-associativity equations, we obtain example of Hom-Poisson algebras.

Example 6.1.6 We consider the 3-dimensional Hom-Lie algebra with basis $\{e_1, e_2, e_3\}$, brackets given by

$$\begin{aligned} [e_1, e_2] &= C_{12}^2 e_2 + C_{12}^3 e_3 \\ [e_1, e_3] &= C_{13}^2 e_2 + C_{13}^3 e_3 \\ [e_2, e_3] &= 0, \end{aligned}$$

and morphism $\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$ in the basis $\{e_1, e_2, e_3\}$. The only multiplications giving a Hom-Poisson algebra are of the form

$$\begin{aligned} e_1 \cdot e_2 &= \lambda e_2 \\ e_1 \cdot e_3 &= \lambda e_3 \\ e_2 \cdot e_3 &= 0 \\ e_i \cdot e_i &= 0 \quad \text{for } i = 1, 2, 3. \end{aligned}$$

Example 6.1.7 Other examples of Hom-Poisson algebras of dimension 3 with basis $\{e_1, e_2, e_3\}$ are given by twisting the following Poisson algebra:

$$e_1 \cdot e_1 = e_2 \qquad \{e_1, e_3\} = a e_2 + b e_3,$$

with all other multiplication and brackets equal to zero.

The morphism α is computed to be multiplicative for the multiplication and the bracket.

[1] actually $\frac{n^2(n+1)}{2}$ unknowns using the commutativity

With $a \neq 0,\ b \neq 0$	With $a \neq 0,\ b = 0$
	$\alpha(e_1) = \alpha_{21}e_2$ $\alpha(e_2) = 0$ $\alpha(e_3) = \alpha_{13}e_1 + \alpha_{23}e_2 + \alpha_{33}e_3$
$\alpha(e_1) = \alpha_{11}e_1 + \alpha_{21}e_2 + \alpha_{31}e_3$ $\alpha(e_2) = \alpha_{11}^2 e_2$ $\alpha(e_3) = -\frac{a}{b}\alpha_{11}^2 e_2$	$\alpha(e_1) = \alpha_{11}e_1 + \alpha_{21}e_2 + \alpha_{31}e_3$ $\alpha(e_2) = \alpha_{11}^2 e_2$ $\alpha(e_3) = \frac{\alpha_{11}\alpha_{33} - \alpha_{11}^2}{\alpha_{31}} e_1 + \alpha_{23}e_2 + \alpha_{33}e_3$
	$\alpha(e_1) = \alpha_{11}e_1 + \alpha_{21}e_2$ $\alpha(e_2) = \alpha_{11}^2 e_2$ $\alpha(e_3) = \alpha_{13}e_1 + \alpha_{23}e_2 + \alpha_{11}e_3$

6.2 1-OPERATION STRUCTURES

We recall here some results on flexible structures described in [MS08] and provide a connection with Hom-Poisson algebras. Algebras with one operation were introduced by Loday and studied by Markl and Remm in [MR06]. The twisted version was studied in [Yau10b] where they are called admissible Hom-Poisson algebras.

6.2.1 Flexibles Hom-algebras

Definition 6.2.1 A Hom-algebra $A = (A, \mu, \alpha)$ is called *flexible* if for any $x, y \in A$

$$\mu(\mu(x,y), \alpha(x)) = \mu(\alpha(x), \mu(y,x))). \tag{6.2.1}$$

Remark 6.2.2 Using the Hom-associator $\mathfrak{as}_{\mu,\alpha}$, the condition (6.2.1) may be written as

$$\mathfrak{as}_{\mu,\alpha}(x,y,x) = 0.$$

Lemma 6.2.3 *Let $A = (A, \mu, \alpha)$ be a Hom-algebra. The following assertions are equivalent*

1. *A is flexible.*
2. *For any $x, y \in A$,* $\mathfrak{as}_{\mu,\alpha}(x,y,x) = 0$.
3. *For any $x, y, z \in A$,* $\mathfrak{as}_{\mu,\alpha}(x,y,z) = -\mathfrak{as}_{\mu,\alpha}(z,y,x)$.

Proof. The equivalence of the first two assertions follows from the definition. The assertion $\mathfrak{as}_{\mu,\alpha}(x-z,y,x-z) = 0$ holds by definition and it is equivalent to $\mathfrak{as}_{\mu,\alpha}(x,y,z) + \mathfrak{as}_{\mu,\alpha}(z,y,x) = 0$ by linearity. □

Corollary 6.2.4 *Any Hom-associative algebra is flexible.*

Let $A = (A, \mu, \alpha)$ be a Hom-algebra, where μ is the multiplication and α a twisting map. We define two new multiplications using μ:

$$\forall\ x, y \in A \quad x \bullet y = \mu(x, y) + \mu(y, x), \quad \{x, y\} = \mu(x, y) - \mu(y, x).$$

We set $A^+ = (A, \bullet, \alpha)$ and $A^- = (A, \{\ ,\ \}, \alpha)$.

Proposition 6.2.5 *A Hom-algebra $A = (A, \mu, \alpha)$ is flexible if and only if*

$$\{\alpha(x), y \bullet z\} = \{x, y\} \bullet \alpha(z) + \alpha(y) \bullet \{x, z\}. \tag{6.2.2}$$

Proof. Let A be a flexible Hom-algebra. By Lemma (6.2.3), this is equivalent to $\mathfrak{as}_{\mu,\alpha}(x, y, z) + \mathfrak{as}_{\mu,\alpha}(z, y, x) = 0$ for any $x, y, z \in A$. This implies

$$\begin{aligned} \mathfrak{as}_{\mu,\alpha}(x, y, z) + \mathfrak{as}_{\mu,\alpha}(z, y, x) + \mathfrak{as}_{\mu,\alpha}(x, z, y) + \\ \mathfrak{as}_{\mu,\alpha}(y, z, x) - \mathfrak{as}_{\mu,\alpha}(y, x, z) - \mathfrak{as}_{\mu,\alpha}(z, x, y) = 0 \end{aligned} \tag{6.2.3}$$

By expansion, the previous relation is equivalent to $\{\alpha(x), y \bullet z\} = \{x, y\} \bullet \alpha(z) + \alpha(y) \bullet \{x, z\}$. Conversely, assume we have the condition (6.2.2) in Proposition. By setting $x = z$ in (6.2.3), one gets $\mathfrak{as}_{\mu,\alpha}(x, y, x) = 0$. Therefore A is flexible. □

6.2.2 Link with Hom-Poisson algebras

Definition 6.2.6 A 1-*operation Hom-Poisson algebra* is a Hom-algebra $(A, \cdot, \alpha)$ satisfying, for any $x, y, z \in A$,

$$3\mathfrak{as}_{\cdot,\alpha}(x, y, z) = (x \cdot z) \cdot \alpha(y) + (y \cdot z) \cdot \alpha(x) - (y \cdot x) \cdot \alpha(z) - (z \cdot x) \cdot \alpha(y). \tag{6.2.4}$$

If α is the identity map, A is called a 1-operation Poisson algebra.

We consider a Hom-algebra $(A, \cdot, \alpha)$. We define two operations $\bullet : A \otimes A \to A$ and $\{\ ,\ \} : A \otimes A \to A$ by

$$\forall\ x, y \in A, \qquad x \bullet y = x \cdot y + y \cdot x, \qquad \{x, y\} = x \cdot y - y \cdot x. \tag{6.2.5}$$

Theorem 6.2.7 *$(A, \bullet, \{\ ,\ \}, \alpha)$ is a Hom-Poisson algebra if and only if $(A, \cdot, \alpha)$ is a 1-operation Hom-Poisson algebra.*

Proof. Suppose that $(A, \bullet, \{\ ,\ \}, \alpha)$ is a Hom-Poisson algebra. Since

$$\forall\ x, y \in A,\ x \cdot y = \frac{1}{2}(\{x, y\} + x \bullet y),$$

we have, by expansion,

$$\mathfrak{as}_{\cdot,\alpha}(x,y,z) = (x\cdot y)\cdot\alpha(z) - \alpha(x)\cdot(y\cdot z) = \frac{1}{4}\{\alpha(y),\{z,x\}\},$$

and, on the other hand, using that the multiplication $\bullet$ is Hom-associative and commutative, and that $\{\ ,\ \}$ is a Hom-Lie bracket,

$$(x\cdot z)\cdot\alpha(y) + (y\cdot z)\cdot\alpha(x) - (y\cdot x)\cdot\alpha(z) - (z\cdot x)\cdot\alpha(y) = \frac{3}{4}\{\alpha(y),\{z,x\}\}.$$

We thus have the equation (6.2.4).

Suppose now that the equation (6.2.4) is verified. We have to show that $\bullet$ is Hom-associative and that $\{\ ,\ \}$ is a Hom-Lie bracket.

Using the relation (6.2.4), we obtain the identities

$$\forall\ x,y,z\in A \quad \mathfrak{as}_{\cdot,\alpha}(x,y,z) + \mathfrak{as}_{\cdot,\alpha}(y,z,x) + \mathfrak{as}_{\cdot,\alpha}(z,x,y) = 0 \tag{6.2.6}$$
$$\forall\ x,y,z\in A \quad \mathfrak{as}_{\cdot,\alpha}(x,y,z) + \mathfrak{as}_{\cdot,\alpha}(z,y,x) = 0. \tag{6.2.7}$$

This last identity (6.2.7) shows that $(A,\cdot,\alpha)$ is a Hom-flexible algebra using Lemma 6.2.3.

We now obtain

$$\begin{aligned}&\mathfrak{as}_{\bullet,\alpha}(x,y,z) = (x\bullet y)\bullet\alpha(z) - \alpha(x)\bullet(y\bullet z)\\ =&\mathfrak{as}_{\cdot,\alpha}(y,z,x) + \mathfrak{as}_{\cdot,\alpha}(x,z,y) - (\mathfrak{as}_{\cdot,\alpha}(z,y,x) + \mathfrak{as}_{\cdot,\alpha}(x,y,z))\\ \overset{(6.2.7)}{=}&0.\end{aligned}$$

So the product $\bullet$ is Hom-associative and commutative by definition. Moreover,

$$\begin{aligned}&J_{\{\ ,\ \},\alpha}(x,y,z) = \{\alpha(x),\{y,z\}\} + \{\alpha(y),\{z,x\}\} + \{\alpha(z),\{x,y\}\}\\ =&-(\mathfrak{as}_{\cdot,\alpha}(x,y,z) + \mathfrak{as}_{\cdot,\alpha}(y,z,x) + \mathfrak{as}_{\cdot,\alpha}(z,x,y)) +\\ &\mathfrak{as}_{\cdot,\alpha}(x,z,y) + \mathfrak{as}_{\cdot,\alpha}(y,x,z) + \mathfrak{as}_{\cdot,\alpha}(z,y,x)\\ \overset{(6.2.6)}{=}&0,\end{aligned}$$

so $\{\ ,\ \}$ is a Hom-Lie bracket. Since A is flexible, Proposition 6.2.5 leads to the compatibility between $\bullet$ and $\{\ ,\ \}$,

$$\{\alpha(x), y\bullet z\} = \{x,y\}\bullet\alpha(z) + \alpha(y)\bullet\{x,z\}.$$

So $(A,\bullet,\{\ ,\ \},\alpha)$ is a Hom-Poisson algebra. □

Proposition 6.2.8 *Let $(A,\cdot)$ be a 1-operation Poisson algebra, and $\alpha : A\to A$ be a linear map multiplicative for the multiplication $\cdot$, i.e. $\alpha\circ\cdot = \cdot\circ\alpha^{\otimes 2}$, then $A_\alpha = (A,\cdot_\alpha = \alpha\circ\cdot,\alpha)$ is a 1-operation Hom-Poisson algebra.*

Proof. We have

$$\begin{aligned}\mathfrak{Jas}_{\cdot_\alpha,\alpha}(x,y,z) &= (x\cdot_\alpha y)\cdot_\alpha \alpha(z)-\alpha(x)\cdot_\alpha (y\cdot_\alpha z)\\ &= \alpha(\alpha(x\cdot y)\cdot\alpha(z))-\alpha(\alpha(x)\cdot\alpha(y\cdot z)) = \alpha^2((x\cdot y)\cdot z - x\cdot(y\cdot z)),\end{aligned}$$

and since $\cdot$ verifies the 1-operation equation,

$$\begin{aligned}\mathfrak{Jas}_{\cdot_\alpha,\alpha}(x,y,z) &= \alpha^2((x\cdot z)\cdot y+(y\cdot z)\cdot x-(y\cdot x)\cdot z-(z\cdot x)\cdot y)\\ &= \alpha(\alpha(x\cdot z)\cdot\alpha(y))+\alpha(\alpha(y\cdot z)\cdot\alpha(x))\\ &\quad -\alpha(\alpha(y\cdot x)\cdot\alpha(z))-\alpha(\alpha(z\cdot x)\cdot\alpha(y))\\ &= (x\cdot_\alpha z)\cdot_\alpha (y)+(y\cdot_\alpha z)\cdot_\alpha \alpha(x)-(y\cdot_\alpha x)\cdot_\alpha \alpha(z)-(z\cdot_\alpha x)\cdot_\alpha \alpha(y).\end{aligned}$$

□

6.3 Hom-coPoisson algebras and duality

In this section, we extend the connection between Lie bialgebras and coPoisson-Hopf algebras presented in [CP94] to the Hom setting. We also extend to Hom-algebras the result stated in [OP11], that the Hopf dual of a coPoisson Hopf algebra is a Poisson-Hopf algebra.

6.3.1 Link with Hom-Lie bialgebras

Definition 6.3.1 A *Hom-coPoisson algebra* consists of a cocommutative coassociative Hom-coalgebra $(A,\Delta,\varepsilon,\alpha)$ equipped with a skew-symmetric linear map $\delta : A \to A\otimes A$, the Hom-coPoisson cobracket, satisfying the following conditions

(i) (Hom-coJacobi identity)

$$(id+\sigma+\sigma^2)\circ(\alpha\otimes\delta)\circ\delta = 0, \qquad (6.3.1)$$

(ii) (Hom-coLeibniz rule)

$$(\Delta\otimes\alpha)\circ\delta = (\alpha\otimes\delta)\circ\Delta+\tau_{23}\circ(\delta\otimes\alpha)\circ\Delta. \qquad (6.3.2)$$

It is denoted by a tuple $(A,\Delta,\varepsilon,\alpha,\delta)$.

Proposition 6.3.2 *Let $(A,\Delta,\varepsilon,\alpha,\delta)$ be a Hom-coPoisson algebra and $\beta : A\to A$ be a Hom-coPoisson algebra morphism. Then $(A,\Delta_\beta=\Delta\circ\beta,\varepsilon,\alpha\circ\beta,\delta_\beta=\delta\circ\beta)$ is a Hom-coPoisson algebra.*

Proof. Theorem 5.2.2 insures that $(A,\Delta_\beta,\varepsilon,\alpha\circ\beta)$ is a coassociative Hom-coalgebra and Theorem 5.3.3 that $(A,\delta_\beta,\alpha\circ\beta)$ is a Hom-Lie coalgebra. It remains to show the compatibility condition (6.3.2). On the left-hand side, we have

$$(\Delta_\beta\otimes\alpha\circ\beta)\circ\delta_\beta=(\Delta\circ\beta\otimes\alpha\circ\beta)\circ\delta\circ\beta=(\Delta\otimes\alpha)\circ\beta^2,$$

and the right-hand side gives

$$\begin{aligned}&(\alpha\circ\beta\otimes\delta_\beta)\circ\Delta_\beta+\tau_{23}\circ(\delta_\beta\otimes\alpha\circ\beta)\circ\Delta_\beta\\&\quad=(\alpha\circ\beta\otimes\delta\circ\beta)\circ\Delta\circ\beta+\tau_{23}\circ(\delta\circ\beta\otimes\alpha\circ\beta)\circ\Delta\circ\beta\\&\quad=[(\alpha\otimes\delta)\circ\Delta+\tau_{23}\circ(\delta\otimes\alpha)\circ\Delta]\circ\beta^2,\end{aligned}$$

which ends the proof. □

We may state the following Corollaries. Starting from a classical coPoisson algebra, we may construct Hom-coPoisson algebras using coPoisson algebra morphisms. On the other hand a Hom-coPoisson algebra gives rise to infinitely many Hom-coPoisson algebras.

Corollary 6.3.3

1. *Let $(A,\Delta,\varepsilon,\delta)$ be a coPoisson algebra and $\beta: A\to A$ be a coPoisson algebra morphism. Then $(A,\Delta_\beta=\Delta\circ\beta,\varepsilon,\beta,\delta_\beta=\delta\circ\beta)$ is a Hom-coPoisson algebra.*

2. *Let $(A,\Delta,\varepsilon,\alpha,\delta)$ be a Hom-coPoisson algebra. Then for any non negative integer n, we have $(A,\Delta\circ\alpha^n,\varepsilon,\alpha^{n+1},\delta\circ\alpha^n)$ is a Hom-coPoisson algebra.*

Definition 6.3.4 A *Hom-coPoisson bialgebra* $(A,\mu,\eta,\Delta,\varepsilon,\alpha,\delta)$ is a Hom-coPoisson algebra $(A,\Delta,\varepsilon,\alpha,\delta)$ which is also a Hom-bialgebra $(A,\mu,\eta,\Delta,\varepsilon,\alpha)$, the two structures being compatible in the sense that δ is a derivation along Δ,

$$\delta\circ\mu=\mu^{[2]}\circ(\delta\otimes\Delta+\Delta\otimes\delta).$$

A *Hom-coPoisson Hopf algebra* $(A,\mu,\eta,\Delta,\varepsilon,S,\alpha,\delta)$ is a Hom-coPoisson bialgebra $(A,\mu,\eta,\Delta,\varepsilon,\alpha,\delta)$ with an antipode S, such that the tuple $(A,\mu,\eta,\Delta,\varepsilon,S,\alpha)$ is a Hom-Hopf algebra.

Theorem 6.3.5 *Let $(\mathfrak{g},[\ ,\ ],\alpha)$ be a Hom-Lie algebra. If its universal enveloping algebra $U_{HLie}(\mathfrak{g})$ has a Hom-coPoisson structure δ such that $\delta\circ\alpha=\alpha^{\otimes 2}\circ\delta$, making it a Hom-coPoisson bialgebra, then $\delta(\mathfrak{g})\subset\mathfrak{g}\otimes\mathfrak{g}$, and $\delta|_{\mathfrak{g}}$ equips $(\mathfrak{g},[\ ,\ ],\alpha,\delta|_{\mathfrak{g}},id)$ with a structure of Hom-Lie bialgebra. Conversely, for a Hom-Lie bialgebra $(\mathfrak{g},[\ ,\ ],\delta,\alpha)$, the cobracket $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ extends uniquely to a Hom-coPoisson cobracket on $U_{HLie}(\mathfrak{g})$, which makes $U_{HLie}(\mathfrak{g})$ a Hom-coPoisson bialgebra.*

Proof. Let $\delta : U_{HLie}(\mathfrak{g}) \to U_{HLie}(\mathfrak{g}) \otimes U_{HLie}(\mathfrak{g})$ be a Hom-coPoisson cobracket on $U_{HLie}(\mathfrak{g})$. To show that $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}$, let $x \in \mathfrak{g}$, and write $\delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$ where $x^{(1)}, x^{(2)} \in U_{HLie}(\mathfrak{g})$. We may assume that the $x^{(2)}$ are linearly independent. By the Hom-coLeibniz condition (6.3.2), we have

$$\begin{aligned}\sum_{(x)} \Delta(x^{(1)}) \otimes \alpha(x^{(2)}) =& \alpha(\mathbf{1}) \otimes \delta(\alpha(x)) + \alpha(\alpha(x)) \otimes \delta(\mathbf{1}) \\ &+ \tau_{23} \circ (\delta(\mathbf{1}) \otimes \alpha(\alpha(x)) + \delta(\alpha(x)) \otimes \alpha(\mathbf{1}))\end{aligned}$$

since $x \in \mathfrak{g}$ and $\Delta(x) = \mathbf{1} \otimes \alpha(x) + \alpha(x) \otimes \mathbf{1}$. Moreover, $\alpha(\mathbf{1}) = \mathbf{1}$ and δ is a derivation along Δ so $\delta(\mathbf{1}) = 0$, hence

$$\begin{aligned}\sum_{(x)} \Delta(x^{(1)}) \otimes \alpha(x^{(2)}) &= \mathbf{1} \otimes \delta(\alpha(x)) + \tau_{23} \circ (\delta(\alpha(x)) \otimes \mathbf{1}) \\ &= \sum_{(x)} \Big(\mathbf{1} \otimes \alpha(x^{(1)}) + \alpha(x^{(1)}) \otimes \mathbf{1}\Big) \otimes \alpha(x^{(2)})\end{aligned}$$

using the multiplicativity $\delta \circ \alpha = \alpha^{\otimes 2} \circ \delta$. It follows that the $x^{(1)}$ are Hom-primitive elements ($\Delta(x) = 1 \otimes \alpha(x) + \alpha(x) \otimes 1$) of $U_{HLie}(\mathfrak{g})$, hence $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes U_{HLie}(\mathfrak{g})$. Since δ is skew-symmetric,

$$\delta(\mathfrak{g}) \subset (\mathfrak{g} \otimes U_{HLie}(\mathfrak{g})) \cap (U_{HLie}(\mathfrak{g}) \otimes \mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}.$$

To prove the compatibility condition (5.3.4) for $\delta|_{\mathfrak{g}}$ and the twisting maps α and id, let $x, y \in \mathfrak{g}$ and compute

$$\begin{aligned}\delta([x,y]) =& \delta(xy - yx) \\ =& \delta(x)\Delta(y) + \Delta(x)\delta(y) - \delta(y)\Delta(x) - \Delta(y)\delta(x) \\ =& [\Delta(x), \delta(y)] - [\Delta(y), \delta(x)] \\ =& [\alpha(x), y^{(1)}] \otimes y^{(2)} + y^{(1)} \otimes [\alpha(x), y^{(2)}] \\ & - [\alpha(y), x^{(1)}] \otimes x^{(2)} - x^{(1)} \otimes [\alpha(y), x^{(2)}] \\ =& ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(y)).\end{aligned}$$

Conversely, $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ uniquely extends in $\overline{\delta} : U_{HLie}(\mathfrak{g}) \to U_{HLie}(\mathfrak{g}) \otimes U_{HLie}(\mathfrak{g})$ such that $\overline{\delta}|_{\mathfrak{g}} = \delta$, with the formula

$$\overline{\delta}(xy) = \overline{\delta}(x)\Delta(y) + \Delta(x)\overline{\delta}(y).$$

This gives $U_{HLie}(\mathfrak{g})$ a structure of Hom-coPoisson bialgebra. □

6.3.2 Duality

Definition 6.3.6 An algebra A over $\mathbb{K}$ is said to be an almost normalizing extension over $\mathbb{K}$ if A is a finitely generated $\mathbb{K}$-algebra with generators $x_1, \ldots, x_n$ satisfying the condition

$$x_i x_j - x_j x_i \in \sum_{l=1}^{n} \mathbb{K} x_l + \mathbb{K} \tag{6.3.3}$$

for all i, j.

Lemma 6.3.7 *Let A be an almost normalizing extension of $\mathbb{K}$ with generators $x_1, \dots, x_n$. Then A is spanned by all standard monomials*

$$x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, \qquad r_i \in \mathbb{N}$$

together with the unity 1.

Proof. This follows immediately from induction on the degree of monomials. □

Recall that finite dual A° of a Hom-bialgebra A consists of

$$A^\circ = \{f \in A^*, f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\},$$

where A^* is the dual vector space of A.

Theorem 6.3.8 *Let A be a Hom-coPoisson bialgebra with Poisson co-bracket δ and twisting map α. If A is an almost normalizing extension over $\mathbb{K}$, then the finite dual A° is a Hom-Poisson bialgebra with twisting map α° and bracket*

$$\{f, g\}(x) = \langle \delta(x), f \otimes g \rangle, \qquad x \in A \tag{6.3.4}$$

for any $f, g \in A^\circ$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between the vector space $A \otimes A$ and its dual vector space.

Proof. The proof is almost the same as in [OP11]. We do not reproduce here the first step showing that the bracket (6.3.4) is well-defined, it uses the fact that A is an almost normalizing extension over $\mathbb{K}$.

The skew-symmetry follows from $\tau_{12} \circ \delta = -\delta$, we have

$$\begin{aligned} \{g, f\}(x) &= \langle \delta(x), g \otimes f \rangle = \langle \tau_{12} \circ \delta, f \otimes g \rangle \\ &= -\langle \delta(x), f \otimes g \rangle = -\{f, g\}(x), \end{aligned}$$

for all $x \in A$.

The equation (6.3.4) satisfies the Hom-Leibniz rule: since

$$\{fg, \alpha^\circ(h)\}(x) = \langle (\Delta \otimes \alpha) \circ \delta(x), f \otimes g \otimes h \rangle$$

and

$$\begin{aligned} &(\alpha^\circ(f)\{g, h\} + \{f, h\}\alpha^\circ(g))(x) \\ &= \langle (\alpha \otimes \delta) \circ \Delta(x), f \otimes g \otimes h \rangle + \langle \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta(x), f \otimes g \otimes h \rangle \end{aligned}$$

for $x \in A$ and $f, g, h \in A^\circ$, it is enough to show that

$$(\Delta \otimes \alpha) \circ \delta = (\alpha \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta,$$

but this is just the Hom-coLeibniz rule for δ.

The equation (6.3.4) satisfies the Hom-Jacobi identity: we have

$$\begin{aligned}
\{\alpha^\circ(f), \{g, h\}\}(x) &= \langle (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle, \\
\{\alpha^\circ(g), \{h, f\}\}(x) &= \langle \sigma \circ (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle, \\
\{\alpha^\circ(h), \{f, g\}\}(x) &= \langle \sigma^2 \circ (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle,
\end{aligned}$$

for $x \in A$ and $f, g, h \in A^\circ$. Hence (6.3.4) satisfies the Hom-Jacobi identity if and only if δ satifies

$$(id + \sigma + \sigma^2) \circ (\alpha \otimes \delta) \circ \delta = 0,$$

which is just the Hom-coJacobi identity of δ.

The bracket defined by (6.3.4) satisfies the compatibility condition with the comultiplication of the Hom-bialgebra, it is a μ-coderivation: since δ is a Δ-derivation, we have for $f, g \in A^\circ$

$$\begin{aligned}
\Delta(\{f, g\})(x \otimes y) = \{f, g\}(xy) &= \langle \delta(xy), f \otimes g \rangle \\
&= \langle \delta(x)\Delta(y), f \otimes g \rangle + \langle \Delta(x)\delta(y), f \otimes g \rangle \\
&= \langle \delta(x), f_1 \otimes g_1 \rangle \langle \Delta(y), f_2 \otimes g_2 \rangle \\
&\quad + \langle \Delta(x), f_1 \otimes g_2 \rangle \langle \delta(y), f_2 \otimes g_2 \rangle \\
&= \{f_1, g_1\}(x)(f_2 g_2)(y) + (f_1 g_1)(x)\{f_2, g_2\}(y) \\
&= \{\Delta(f), \Delta(g)\}(x \otimes y).
\end{aligned}$$

Finally, the bracket defined by (6.3.4) equips A° with the structure of a Hom-Poisson bialgebra, the twisting map being α°. □

Let $(\mathfrak{g}, [\ ,\ ], \delta, \alpha)$ be a Hom-Lie bialgebra, $U_{HLie}(\mathfrak{g})$ the universal enveloping Hom-bialgebra of $\mathfrak{g}$ with comultiplication Δ. The cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is extended uniquely to a Δ-derivation $\overline{\delta} : U_{HLie}(\mathfrak{g}) \to U_{HLie}(\mathfrak{g}) \otimes U_{HLie}(\mathfrak{g})$ such that $\overline{\delta}|_{\mathfrak{g}} = \delta$ and $\overline{\delta}(xy) = \overline{\delta}(x)\Delta(y) + \Delta(x)\overline{\delta}(y)$. Then $U_{HLie}(\mathfrak{g})$ is a Hom-coPoisson bialgebra with cobracket $\overline{\delta}$.

Corollary 6.3.9 *Let $(\mathfrak{g}, [\ ,\ ], \delta, \alpha)$ be a finite dimensional Hom-Lie bialgebra. Then the dual $U_{HLie}(\mathfrak{g})^\circ$ of the universal enveloping Hom-bialgebra $U_{HLie}(\mathfrak{g})$ is a Hom-Poisson bialgebra with Poisson bracket*

$$\{f, g\}(x) = \langle \overline{\delta}(x), f \otimes g \rangle, \qquad x \in U_{HLie}(\mathfrak{g})$$

for $f, g \in U_{HLie}(\mathfrak{g})^\circ$.

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$. Then $U_{HLie}(\mathfrak{g})$ is an almost normalizing extension over $\mathbb{K}$ with generators $x_1, \ldots, x_n$. Thus the result follows from Theorem 6.3.8. □

7

Deformation and quantization of Hom-algebras

Contents

The theory of formal deformation can be extended to Hom-algebraic structures. We give the definitions for Hom-algebras, Hom-coalgebras and Hom-bialgebras, as well as some results generalising the classical case. We then seek the Poisson morphisms and morphisms of the ⋆-product of Moyal-Weyl, to obtain Hom-Poisson and Hom-associative algebras by twist.

7.1 Formal Hom-deformation

Let A be a $\mathbb{K}$ vector space. Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{K}$ and $A[[t]]$ be the set of formal power series whose coefficients are elements of A, ($A[[t]]$ is obtained by extending the coefficients domain of A from $\mathbb{K}$ to $\mathbb{K}[[t]]$). Then $A[[t]]$ is a $\mathbb{K}[[t]]$-module. When A is finite-dimensional, we have $A[[t]] = A \otimes \mathbb{K}[[t]]$. Note that A is a submodule of $A[[t]]$. We define formal deformations for Hom-associative algebras, Hom-coassociative coalgebras and Hom-bialgebras.

7.1.1 Formal deformation of Hom-associative algebras

Definition 7.1.1 Let $A = (A, \mu_0, \alpha_0)$ be a Hom-associative algebra. A formal Hom-associative deformation of A is given by a $\mathbb{K}[[t]]$-bilinear map $\mu_t : A[[t]] \otimes A[[t]] \to A[[t]]$ and a $\mathbb{K}[[t]]$-linear map $\alpha_t : A[[t]] \to A[[t]]$ of the form

$$\mu_t = \sum_{i \geqslant 0} \mu_i t^i \qquad \text{and} \qquad \alpha_t = \sum_{i \geqslant 0} \alpha_i t^i \tag{7.1.1}$$

where each μ_i is a $\mathbb{K}$-bilinear map $\mu_i : A \otimes A \to A$ (extended to be $\mathbb{K}[[t]]$-bilinear) and each α_i is a $\mathbb{K}$-linear map $\alpha_i : A \to A$ (extended to be $\mathbb{K}[[t]]$-linear) such that the following formal Hom-associativity condition holds:

$$\mathfrak{as}_{\mu_t, \alpha_t} = \mu_t \circ (\mu_t \otimes \alpha_t - \alpha_t \otimes \mu_t) = 0. \tag{7.1.2}$$

If $\alpha_t = id$ the definition reduces to formal deformation of an associative algebra already presented in Section 1.2.4.

Example 7.1.2 We can consider the Jackson Hom-Lie algebra $\mathfrak{sl}(2)$ from Example 5.1.4 as a formal deformation of $\mathfrak{sl}(2)$. Setting $q = 1 + t$, the brackets and twist map are given by

$$\begin{aligned} [h, e]_t &= 2e, \\ [h, f]_t &= -2f - 2ft, \\ [e, f]_t &= h + \frac{h}{2}t, \end{aligned} \qquad \begin{aligned} \alpha_t(e) &= e + et, \\ \alpha_t(f) &= f + 2ft + ft^2, \\ \alpha_t(h) &= h + ht, \end{aligned}$$

The coefficients of the powers of t are the following.

$$[h, e]_0 = 2e, \quad [h, f]_0 = -2f, \quad [e, f]_0 = h,$$
$$\alpha_0(e) = e \quad \alpha_0(f) = f \quad \alpha_0(h) = h$$

$$[h, e]_1 = 0, \quad [h, f]_1 = -2f, \quad [e, f]_1 = \frac{h}{2},$$
$$\alpha_1(e) = e \quad \alpha_1(f) = 2f \quad \alpha_1(h) = h$$

$$\alpha_2(f) = f$$

We recover the classical bracket of $\mathfrak{sl}(2)$ at order 0 and $\alpha_0 = id$. There are other possibles twisting morphisms (given in Example 5.1.11) for the same bracket $[\ ,\ ]_t$, for example

$$\alpha_t(e) = e, \quad \alpha_t(f) = \frac{2+t}{2(1+t)}f = f + \sum_{k=0}^{\infty} \frac{(-1)^k f}{2} t^k, \quad \alpha_t(h) = h + \frac{h}{2}t.$$

The equation (7.1.2) can be written

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \left(\mu_i(\alpha_k(x),\mu_j(y,z)) - \mu_i(\mu_j(x,y,\alpha_k(z)))\right) t^{i+j+k} = 0. \tag{7.1.3}$$

Introducing the α-associators

$$(x,y,z) \mapsto \mu_i \circ_\alpha \mu_j(x,y,z) := \mu_i(\alpha(x),\mu_j(y,z)) - \mu_i(\mu_j(x,y,\alpha(z)),$$

the deformation equation may be written as follows

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}(\mu_i \circ_{\alpha_k} \mu_j) t^{i+j+k} = 0 \qquad \text{or} \qquad \sum_{s\in\mathbb{N}} t^s \sum_{k=0}^{s}\sum_{i=0}^{s-k} \mu_i \circ_{\alpha_k} \mu_{s-k-i} = 0.$$

It is equivalent to the infinite system:

$$\sum_{k=0}^{s}\sum_{i=0}^{s-k} \mu_i \circ_{\alpha_k} \mu_{s-k-i} = 0, \qquad s \in \mathbb{N}. \tag{7.1.4}$$

The A-valued Hochschild type cohomology of Hom-associative algebras initiated in [MS10b] and extended in [AEM11] suits and leads to the following cohomological interpretation:

1. There is a natural bijection between $H^2(A,A)$ and the set of equivalence classes of deformation (mod t^2) of A.

2. If $H^2(A,A) = 0$ then every deformation of A is trivial.

The fact that the antisymmetrization of the first order element of a deformation of an associative algebra defines a Poisson bracket remains true in the Hom setting. More precisely, we have the following theorem.

Theorem 7.1.3 ([MS10b, Theorem 4.9]) *Let $A = (A,\mu_0,\alpha)$ be a commutative Hom-associative algebra and $A_t = (A,\mu_t,\alpha_t)$ be a deformation of A. Consider the bracket defined for $x,y \in A$ by $\{x,y\} = \mu_1(x,y) - \mu_1(y,x)$ where μ_1 is the first order element of the deformation μ_t. Then $(A,\mu_0,\{\ ,\ \},\alpha_0)$ is a Hom-Poisson algebra.*

The proof is mainly computational, it leans on the properties of the α-associators, and on rewriting the deformation equations (7.1.4) in terms of coboundary operators.

7.1.2 Deformations of Hom-coalgebras and Hom-Bialgebras

Definition 7.1.4 Let (A,Δ,α) be a Hom-coalgebra. A formal Hom-coalgebra deformation of A is given by linear maps $\Delta_t : A[[t]] \to A[[t]]\otimes A[[t]]$ and $\alpha_t : A[[t]] \to A[[t]]$ such that $\Delta_t = \sum_{i\geqslant 0}\Delta_i t^i$ and $\alpha_t = \sum_{i\geqslant 0}\alpha_i t^i$ where each Δ_i, α_i are linear maps $\Delta_i : A \to A\otimes A$ and $\alpha_i : A \to A$ (extended to be $\mathbb{K}[[t]]$-linear) such that the following formal Hom-coassociativity condition holds:

$$(\Delta_t \otimes \alpha_t - \alpha_t \otimes \Delta_t)\circ \Delta_t = 0. \tag{7.1.5}$$

Definition 7.1.5 Let (A,μ,Δ,α) be a Hom-bialgebra. A formal Hom-bialgebra deformation of A is given by linear maps $\mu_t : A[[t]]\otimes A[[t]] \to A[[t]]$, $\Delta_t : A[[t]] \to A[[t]]\otimes A[[t]]$ and $\alpha_t : A[[t]] \to A[[t]]$ such that

1. $(A[[t]],\mu_t,\alpha_t)$ is a Hom-associative algebra,
2. $(A[[t]],\Delta_t,\alpha_t)$ is a Hom-coassociative coalgebra,
3. the multiplication and the comultiplication are compatible, that is
$$\Delta_t \circ \mu_t = (\mu_t \otimes \mu_t)\circ \tau_{23}\circ (\Delta_t \otimes \Delta_t).$$

It is shown in [DM] that deformations are controlled by Hochschild type cohomology and any deformation of an unital Hom-associative algebra (resp. counital Hom-coassociative colgebra) is equivalent to an unital Hom-associative algebra (resp. counital Hom-coassociative colgebra). Furthermore, any deformation of a Hom-Hopf algebra as a Hom-bialgebra is automatically a Hom-Hopf algebra.

In a similar way as for Hom-associative algebra, we have the following result.

Theorem 7.1.6 *Let $A = (A,\Delta_0,\alpha)$ be a cocommutative Hom-coassociative coalgebra and $A_t = (A,\Delta_t,\alpha_t)$ be a deformation of A. Consider the cobracket defined for $x \in A$ by $\delta(x) = \Delta_1(x) - \Delta_1^{op}(x)$ where Δ_1 is the first order element of the deformation Δ_t. Then $(A,\Delta_0,\delta,\alpha_0)$ is a Hom-coPoisson algebra.*

7.2 Quantization and twisting of ★-products

We set the problem of quantization by deformation for Hom-associative algebras.

Let $(A,\cdot,\{\ ,\ \},\alpha)$ be a commutative Hom-associative algebra endowed with a Hom-Poisson bracket $\{\ ,\ \}$.

Definition 7.2.1 A $\star$-product on A is a one parameter formal deformation defined on A by

$$f \star_t g = \sum_{r=0}^{\infty} t^r \mu_r(f,g)$$

such that

1. the $\star$-product in $A[[t]]$ is Hom-associative, that is

$$\forall r \in \mathbb{N}, \quad \sum_{i=0}^{r} (\mu_i(\mu_{r-i}(f,g), \alpha(h)) - (\mu_i(\alpha(f), \mu_{r-i}(g,h))) = 0,$$

2. $\mu_0(f,g) = f \cdot g$,
3. $\mu_1(f,g) - \mu_1(g,f) = \{f,g\}$,
4. $\mu_r(f,1) = \mu_r(1,f) = 0 \quad \forall\, r > 0.$

Remark 7.2.2

- The condition 2. shows that $[f,g] := \frac{1}{t}(f \star_t g - g \star_t f)$ is a deformation of the Hom-Lie structure $\{\ ,\ \}$.
- The condition $\mu_1(f,g) - \mu_1(g,f) = \{f,g\}$ expresses the correspondence between the deformation and the Hom-Poisson structure

$$\frac{f \star_t g - g \star_t f}{t}|_{t=0} = \{f,g\}.$$

Similarly we set the dual version of the quantization problem as follows.

Let A be cocommutative Hom-coPoisson bialgebra (resp. Hopf algebra) and let δ be its Poisson cobracket. A quantization of A is a Hom-bialgebra (resp. Hom-Hopf algebra) deformation A_t of A such that

$$\delta(x) = \frac{\Delta_t(a) - \Delta_t^{op}(a)}{t} \pmod{t},$$

where $x \in A$ and a is any element of $A[[t]]$ such that $x = a \pmod{t}$.

Theorem 7.2.3 *Let $(A,\star)$ be an associative deformation of an associative algebra (A,μ_0), with $\star = \sum_{i\in\mathbb{N}} \mu_i t^i$. Let $\alpha : A \to A$ be a morphism such that for all $i \in \mathbb{N}$, $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$. Then $(A, \star_\alpha = \alpha \circ \star, \alpha)$ is a Hom-associative deformation of $(A, \mu_{0,\alpha}, \alpha)$.*

Proof. Since for all $i \in \mathbb{N}$, $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$, we also have $\alpha \circ \star = \star \circ \alpha^{\otimes 2}$. Thus α is an algebra morphism of $(A,\star)$ which endows $(A,\star_\alpha,\alpha)$ of a Hom-associative structure by the twisting principle Theorem 5.1.5. The first term of this Hom-associative deformation is $\mu_{0,\alpha}$ which is also Hom-associative. □

7.2.1 Twists of Moyal-Weyl $\star$-product

In the following, we twist the Moyal-Weyl product. It is the associative $\star$-product corresponding to the deformation of the Poisson phase-space bracket, one of the first examples of Kontsevich formal deformation [Kon03].

We consider the Poisson algebra of polynomials of two variables $\mathsf{R} = (\mathbb{R}[x,y], \cdot, \{\ ,\ \})$ where the Poisson bracket of two polynomials is given by $\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$.

The associated associative algebra is $(\mathbb{R}[x,y][[\lambda]], \star_{MW})$ where the $\star$-product is given by the Moyal-Weyl formula

$$\star_{MW} = \mu_0 \circ e^{\frac{\lambda}{2}(\partial_x \otimes \partial_y - \partial_y \otimes \partial_x)} \tag{7.2.1}$$

which, expanding the exponentials writes on elements $f,g \in \mathbb{R}[x,y]$ as

$$f \star_{MW} g = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} (\partial_x^k \partial_y^{n-k} f)(\partial_x^{n-k}\partial_y^k g), \tag{7.2.2}$$

noting λ the formal parameter.

This $\star$-product is equivalent to the $\star$-product which writes more simply $\star = \mu_0 \circ e^{\lambda \partial_x \otimes \partial_y}$, on elements, we obtain

$$f \star g = \sum_{n \in \mathbb{N}} \frac{\partial^n f}{\partial x^n}\frac{\partial^n g}{\partial y^n}\frac{\lambda^n}{n!} = \sum_{n\in\mathbb{N}} \mu_n(f,g)\lambda^n, \tag{7.2.3}$$

where $\mu_n(f,g) = \frac{1}{n!}\frac{\partial^n f}{\partial x^n}\frac{\partial^n g}{\partial y^n}$.

The isomorphism between the associative algebras $(\mathbb{R}[x,y][[\lambda]], \star_{MW})$ and $(\mathbb{R}[x,y][[\lambda]], \star)$ is given by

$$S = e^{-\frac{\lambda}{2}\partial_x\partial_y} \qquad S \circ \star = \star_{MW} \circ (S \otimes S).$$

Proposition 7.2.4 *A morphism $\alpha : \mathbb{R}[x,y][[\lambda]] \to \mathbb{R}[x,y][[\lambda]]$ satisfying $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$ for all $i \in \mathbb{N}$ which gives $(\mathbb{R}[x,y],\ \star_\alpha = \alpha\star, \alpha)$ a structure of Hom-associative algebra is of the form*

$$\alpha(x) = ax + b \text{ and } \alpha(y) = \frac{1}{a}y + c \qquad \text{where} \quad a,b,c \in \mathbb{R},\ a \neq 0. \tag{7.2.4}$$

Proof. Let $\alpha : \mathbb{R}[x,y] \to \mathbb{R}[x,y]$ be a morphism such that for all $i \in \mathbb{N}$, $\alpha\mu_i = \mu_i\alpha^{\otimes 2}$. In particular, for $i = 0$,

$$\alpha(fg) = \alpha(f)\alpha(g), \tag{7.2.5}$$

which shows that α is multiplicative, so it is sufficient to define it on x and y. For $i = 1$, we get

$$\alpha\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right)=\frac{\partial\alpha(f)}{\partial x}\frac{\partial\alpha(g)}{\partial y}, \tag{7.2.6}$$

which implies that $\alpha(\{f,g\})=\{\alpha(f),\alpha(g)\}$.

We set $P_1(x,y):=\alpha(x)$ and $P_2(x,y):=\alpha(y)$. For $f(x,y)=x$ and $g(x,y)=y$, the equation (7.2.6) gives

$$1=\alpha(1)=\frac{\partial P_1}{\partial x}\frac{\partial P_2}{\partial y},$$

so we must have $P_1(x,y)=ax+Q_1(y)$ and $P_2(x,y)=\frac{1}{a}y+Q_2(x)$ with $a\in\mathbb{R}\setminus\{0\}$ and $Q_1,Q_2\in\mathbb{R}[x,y]$. For $f(x,y)=y$ and $g(x,y)=x$, the equation (7.2.6) gives

$$0=\alpha(0)=\frac{\partial P_2}{\partial x}\frac{\partial P_1}{\partial y}=Q'_{2,x}Q'_{1,y}.$$

So we can suppose that $Q_1(y)=b$ is constant. Reporting in the equation (7.2.6), with $f(x,y)=g(x,y)=y$, we find

$$0=\alpha(0)=\frac{\partial P_2}{\partial x}\frac{\partial P_2}{\partial y}=Q'_{2,x}\frac{1}{a},$$

so $Q_2(x)=c$ is constant. It remains to verify that for $i>1$, $\alpha\mu_i=\mu_i\alpha^{\otimes 2}$ *i.e.* for $f,g\in\mathbb{R}[x,y]$, $\alpha\left(\frac{\partial^i f}{\partial x^i}\frac{\partial^i g}{\partial y^i}\frac{1}{i!}\right)=\frac{\partial^i\alpha(f)}{\partial x^i}\frac{\partial^i\alpha(g)}{\partial y^i}\frac{1}{i!}$. By multiplicavity of α, the only non trivial case is $f(x,y)=x^n$ and $g(x,y)=y^m$. We have

$$\alpha\left(\frac{\partial^i f}{\partial x^i}\frac{\partial^i g}{\partial y^i}\frac{1}{i!}\right)=\alpha\left(i!\binom{n}{i}x^{n-i}i!\binom{m}{i}y^{m-i}\frac{1}{i!}\right)=i!\binom{n}{i}i!\binom{m}{i}(ax+b)^{n-i}\left(\frac{1}{a}y+c\right)^{m-i}\frac{1}{i!}$$
$$=i!\binom{n}{i}i!\binom{m}{i}(ax+b)^{n-i}a^i\left(\frac{1}{a}y+c\right)^{m-i}\left(\frac{1}{a}\right)^i\frac{1}{i!}=\frac{\partial^i(ax+b)^n}{\partial x^i}\frac{\partial^i\left(\frac{1}{a}y+c\right)^m}{\partial y^i}\frac{1}{i!}$$
$$=\frac{\partial^i\alpha(f)}{\partial x^i}\frac{\partial^i\alpha(g)}{\partial y^i}\frac{1}{i!}.$$

□

The Hom-algebra $(\mathbb{R}[x,y][[\lambda]],\star_\alpha,\alpha)$ is Hom-associative and not associative if $\alpha\neq id$. Indeed, for $f(x,y)=1$, $g(x,y)=y$ and $h(x,y)=x$, we have

$$(f\star_\alpha g)\star_\alpha h=\alpha(\alpha(f)\star\alpha(g))\star\alpha(h)$$
$$=\alpha\left(1\star\frac{1}{a}y+c\right)\star(ax+b)=\alpha\left(\frac{1}{a}y+c\right)\star(ax+b),$$

and

$$\begin{aligned} f \star_\alpha (g \star_\alpha h) &= \alpha(\alpha(f)) \star \alpha(\alpha(g) \star \alpha(h)) \\ &= 1 \star \alpha\left(\left(\frac{1}{a}y + c\right) \star (ax+b)\right) = \alpha\left(\frac{1}{a}y + c\right) \star \alpha(ax+b) \end{aligned}$$

which are different in general.

More generally, we can consider the Poisson algebra of polynomials of $n \geqslant 3$ variables $(\mathbb{R}[x_1,\ldots,x_n],\cdot,\{\ ,\ \})$ where the Poisson bracket of two polynomials is given by $\{f,g\} = \sum_{1\leqslant i,j\leqslant n} \tau_{ij} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$, with $\tau = (\tau_{ij})$ an antisymmetric $n \times n$ real matrix.

The associated associative algebra is $(\mathbb{R}[x_1 \ldots, x_n][[\lambda]], \star)$ where the $\star$-product is given by

$$f \star g = \sum_{n\in\mathbb{N}} \sum_{1\leqslant i_1,j_1,\ldots,i_n,j_n\leqslant n} \sigma_{i_1 j_1} \cdots \sigma_{i_n j_n} \frac{\partial^n f}{\partial x_{i_1}\cdots \partial x_{i_n}} \frac{\partial^n g}{\partial x_{j_1}\cdots \partial x_{j_n}} \frac{\lambda^n}{n!}, \tag{7.2.7}$$

where $\sigma = (\sigma_{ij})$ is the matrix whose antisymmetrization is τ. For $n \geqslant 3$, set $\mu_n = \frac{1}{n!}\sum_{1\leqslant i_1,j_1,\ldots,i_n,j_n\leqslant n} \sigma_{i_1 j_1} \cdots \sigma_{i_n j_n} \frac{\partial^n f}{\partial x_{i_1}\cdots \partial x_{i_n}} \frac{\partial^n g}{\partial x_{j_1}\cdots \partial x_{j_n}}$.

Proposition 7.2.5 *A morphism $\alpha : \mathbb{R}[x_1,\ldots,x_n] \to \mathbb{R}[x_1,\ldots,x_n]$ satisfying $\alpha\circ\mu_i = \mu_i\circ\alpha^{\otimes 2}$ for all $i \in \mathbb{N}$ which gives $(\mathbb{R}[x_1,\ldots,x_n], \star_\alpha = \alpha\star, \alpha)$ a structure of Hom-associative algebra is of the form*

$$\forall\ 1 \leqslant i \leqslant n,\ \alpha(x_i) = x_i + b_i \quad \text{or} \quad \forall\ 1 \leqslant i \leqslant n,\ \alpha(x_i) = -x_i + b_i, \tag{7.2.8}$$

where $b_i \in \mathbb{R}$.

Proof. The proof is similar to the case with two variables, we get $\alpha(x_i) = a_i x_i + b_i$, except that this time, $a_i a_j = 1$ for all $i \neq j$, which gives the two cases of the proposition. □

To obtain a Hom-associative algebra from $(\mathbb{R}[x,y][[\lambda]], \star)$, the condition $\forall\, i \in \mathbb{N}$, $\alpha\circ\mu_i = \mu_i \circ \alpha^{\otimes 2}$ is very strong. We will see how to construct other morphisms in order to get an Hom-associative algebra by twist. The construction is done by quantization of Poisson morphisms of R.

7.2.2 Twists of the Poisson bracket

We consider again the Poisson algebra of polynomials of two variables $(\mathbb{R}[x,y],\cdot,\{\ ,\ \})$.

In order to construct a Hom-Poisson algebra $(\mathbb{R}[x,y], \cdot_\alpha = \alpha \circ \cdot, \{\ ,\ \}_\alpha = \alpha \circ \{\ ,\ \}, \alpha)$ using the twisting principle Theorem 6.1.4, we need a Poisson morphism $\alpha : \mathbb{R}[x,y] \to \mathbb{R}[x,y]$.

Such a morphism preserve in particular the point-wise multiplication and so is entirely defined by its images on x and y. Denoting $\alpha(x) =: \phi_1(x,y)$, $\alpha(y) =: \phi_2(x,y)$ where $\phi_i \in \mathbb{R}[x,y]$, applying the Poisson morphism α amounts to precompose by a polynomial map $\phi = (\phi_1, \phi_2) : \mathbb{R}^2 \to \mathbb{R}^2$, *i.e.* $\alpha(f) = \phi^*(f) = f \circ \phi$.

Proposition 7.2.6 *The map $\phi^* : \mathbb{R}[x,y] \to \mathbb{R}[x,y]$ is a Poisson algebra morphism if and only if*

$$J_\phi = \begin{vmatrix} \partial_x\phi_1 & \partial_y\phi_1 \\ \partial_x\phi_2 & \partial_y\phi_2 \end{vmatrix} = \partial_x\phi_1\partial_y\phi_2 - \partial_x\phi_2\partial_y\phi_1 = 1 \tag{7.2.9}$$

where J_ϕ is the jacobian determinant of ϕ.

Proof. The map ϕ^* is defined by $\phi = (\phi_1, \phi_2) : \mathbb{R}^2 \to \mathbb{R}^2$. Let $f, g \in \mathbb{R}[x,y]$. By definition, we have

$$(f \cdot g) \circ \phi = f \circ \phi \cdot g \circ \phi \Leftrightarrow \phi^*(f \cdot g) = \phi^*(f) \cdot \phi^*(g).$$

By linearity, we may assume now that $f(x,y) = ax^k y^l$ and $g(x,y) = bx^p y^q$, with $a, b \in \mathbb{R}$, $k, l, p, q \in \mathbb{N}$. We have

$$\begin{aligned}\{f \circ \phi, g \circ \phi\} &= \partial_x(a\phi_1^k\phi_2^l)\partial_y(b\phi_1^p\phi_2^q) - \partial_y(a\phi_1^k\phi_2^l)\partial_x(b\phi_1^p\phi_2^q) \\ &= ab(kq - lp)\phi_1^{k+p-1}\phi_2^{l+q-1}(\partial_x\phi_1\partial_y\phi_2 - \partial_x\phi_2\partial_y\phi_1)\end{aligned}$$

and

$$\begin{aligned}\{f, g\} \circ \phi &= \left(\partial_x(ax^k y^l)\partial_y(bx^p y^q) - \partial_y(ax^k y^l)\partial_x(bx^p y^q)\right) \circ \phi \\ &= ab(kq - lp)\phi_1^{k+p-1}\phi_2^{l+q-1}.\end{aligned}$$

By identification, ϕ^* is a Poisson algebra morphism if and only if $J_\phi = 1$. □

Polynomial automorphisms of $\mathbb{R}^2$ are known, those of constant jacobian 1 are generated by triangular transformations[1].

Theorem 7.2.7 (Automorphisms theorem) *Let $\mathbb{K}$ be a field. The group of polynomial automorphisms* $\mathrm{Aut}[\mathbb{K}^2]$ *is generated by affine transformations $(x,y) \mapsto (a_1x + b_1y + c_1, a_2x + b_2y + c_2)$ where $a_i, b_i, c_i \in \mathbb{K}$ with $a_1b_2 - a_2b_1 \neq 0$, and by triangular transformations $(x,y) \mapsto (x, y + p(x))$ where $p \in \mathbb{K}[x]$.*

However, if we don't assume that the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective, equation $J_\phi = 1$ don't allows yet to conclude that ϕ is invertible. On $\mathbb{K} = \mathbb{C}$, this is the jacobian conjecture, which still is an

[1] since their jacobian matrix is triangular

open question even for $n = 2$.

Jacobian conjecture
A polynomial map of $\mathbb{C}^n$ whose jacobian is a is a non-zero constant is an automorphism of $\mathbb{C}^n$.

The automorphisms theorem was discovered by Jung in 1942 for fields of characteristic 0 and extended by Van der Kulk for any field. Various proofs were proposed, a simple proof on $\mathbb{C}$ and references are given in article [VC03], which also mention references on the work on the jacobian conjecture.

Corollary 7.2.8 *The Poisson automorphisms group of* $\mathsf{R} = (\mathbb{R}[x,y],\cdot,\{\ ,\ \})$ *is generated by lower triangular transformations* $(x,y) \mapsto (x, y + p(x))$ *and upper triangular transformations* $(x,y) \mapsto (x + q(y), y)$*, where* $p \in \mathbb{K}[x]$*,* $q \in \mathbb{K}[y]$*.*

Proof. Indeed, we set in matrix form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

the linear transformations, and

$$\begin{pmatrix} 1 & q(y)/y \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + q(y) \\ y \end{pmatrix} \quad \text{et} \quad \begin{pmatrix} 1 & 0 \\ p(x)/x & 1 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + p(x) \end{pmatrix}$$

the upper and lower triangular transformations. Poisson automorphisms being of jacobian 1, for an automorphism which is a product of linear and triangular transformations, the linear transformations must be of jacobian 1 too, since it is already the case for the triangular ones. Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}), \det M = ad - bc = 1 \right\}.$$

Since $\mathrm{SL}(2,\mathbb{R})$ is generated by transvections, which are exactly the matrix of the triangular morphisms with a polynomial p or q which is linear, the result follows. □

7.2.3 Quantization of the Poisson automorphisms

In this section, we set forth the way to quantize Poisson automorphisms of R "à la Fedosov", (see [Fed96]), to obtain morphisms of the associative Moyal-Weyl algebra.

Morphisms and flows

Let $\phi(x,y) = (x, y + p(x))$ be a triangular morphism, with $p \in \mathbb{R}[x]$. Up to add a parameter t, we can assume that $\phi_t(x,y) = (x, y+tp(x))$ is the flow of the hamiltonian equation

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y}(x,y) = 0 \\ \dot{y} = -\dfrac{\partial H}{\partial x}(x,y) = p(x) \end{cases} \Leftrightarrow \begin{cases} x(t) = x_0 \\ y(t) = y_0 + tp(x) \end{cases} \tag{7.2.10}$$

with $H(x,y) = -\int p(x)dx$ a primitive of $-p$. So $\phi_0(x_0,y_0) = (x_0,y_0)$ is the identity of $\mathbb{R}^2$ and $\phi_t(x_0,y_0) = (x(t),y(t)) = (x_0, y_0 + tp(x_0))$ is the solution at time t.

Differentiating with respect to t, we have

$$\frac{d}{dt}\phi_t(x,y) = (0, p(x)) = X_H(x(t),y(t)) = X_H \circ \phi_t(x,y)$$

where $X_H = \begin{pmatrix} \dfrac{\partial H}{\partial y} & -\dfrac{\partial H}{\partial x} \end{pmatrix}$ is the Lie derivative of H.

As a function of time $\phi_t : \mathbb{R} \to \mathbb{R}^2$, the flow therefore satisfies the differential equation

$$\begin{cases} \dfrac{d}{dt}\phi_t = X_H \circ \phi_t \\ \phi_0 = id_{\mathbb{R}^2} \end{cases}. \tag{7.2.11}$$

We obtain the same equation for the flow of a triangular morphism $(x,y) \mapsto (x + tq(y), y)$ with $q \in \mathbb{R}[y]$.

Such a differential formal equation is a way to concisely write the hidden recursion satisfied by the coefficients of the solution power series. A differential formal equation of this type admits an unique solution (see [Che61, Theorem 1.1]) given by iterated integrations

$$\phi_t = \left(\sum_{k\in\mathbb{N}} Q_k(t)\right)\phi_0 \quad \text{with} \quad Q_{k+1}(t) = \int_0^t X_H(s)Q_k(s)ds \quad \text{and} \quad Q_0 = id \tag{7.2.12}$$

Moreover, if $X_H(t) = X_H$ is constant in time as it is the case here, we get

$$\phi_t = e^{tX_H}\phi_0. \tag{7.2.13}$$

Since we work in the polynomial area with successive compositions of differentials operators, the power series are finite sums when evaluated on polynomials.

Let $\phi_t^*(f) = f \circ \phi_t$ be the Poisson algebra morphism associated to the flow, we have

$$\frac{d}{dt}\phi_t^*(f) = \frac{d}{dt}(f \circ \phi_t) = \begin{pmatrix} \frac{\partial f}{\partial x} \circ \phi_t & \frac{\partial f}{\partial y} \circ \phi_t \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial y} \circ \phi_t \\ -\frac{\partial H}{\partial x} \circ \phi_t \end{pmatrix} = \{f, H\} \circ \phi_t,$$

so the Poisson morphism ϕ_t^* is well-defined by

$$\begin{cases} \frac{d}{dt}\phi_t^* = -\phi_t^* \circ P_H \\ \phi_0^* = id_{\mathsf{R}} \end{cases} \quad \text{where} \quad P_H(f) = \{H, f\}. \tag{7.2.14}$$

More generally, for a Poisson automorphism $\psi_t^* = \phi_t^{1^*} \circ \cdots \circ \phi_t^{n*}$ where the $\phi_t^{i^*}$ are triangular morphisms, we obtain

$$\begin{aligned} \frac{d}{dt}\psi_t^* &= \sum_{k=1}^{n} \phi_t^{1^*} \circ \cdots \circ \phi_t^{k-1^*} \circ \frac{d}{dt}\phi_t^{k^*} \circ \phi_t^{k+1^*} \cdots \circ \phi_t^{n*} \\ &= -\sum_{k=1}^{n} \phi_t^{1^*} \circ \cdots \circ \phi_t^{k-1^*} \circ P_{H_k} \circ \phi_t^{k+1^*} \cdots \circ \phi_t^{n*} \\ &= -\psi_t^* \circ \sum_{k=1}^{n} \left(\phi_t^{k+1^*} \cdots \circ \phi_t^{n*}\right)^{-1} \circ P_{H_k} \circ \underbrace{\phi_t^{k+1^*} \cdots \circ \phi_t^{n*}}_{\phi_t^{>k^*}} \end{aligned}$$

where $P_{H_k}(f) = \{H_k, f\}$ with the various hamiltonians H_k corresponding to the triangular morphisms ϕ_k. Since

$$\begin{aligned} \left(\left(\phi_t^{>k^*}\right)^{-1} \circ P_{H_k} \circ \phi_t^{>k^*}\right)(f) &= \left(\phi_t^{>k^*}\right)^{-1} \left(\left\{H_k, \phi_t^{>k^*}(f)\right\}\right) \\ &= \left\{\left(\phi_t^{>k^*}\right)^{-1}(H_k), f\right\} \\ &=: P_{H_t^k}(f), \end{aligned}$$

setting $P_{H_t} = \sum_{k=1}^{n} P_{H_t^k}$, the Poisson morphism ψ_t^* is well-defined by

$$\begin{cases} \frac{d}{dt}\psi_t^* = -\psi_t^* \circ P_{H_t} \\ \psi_0^* = id_{\mathsf{R}} \end{cases}, \tag{7.2.15}$$

here the operator P_{H_t} depend on the time t.

We can recover the fact that ψ_t^* is a morphism from R by differentiation. Let

$$A_t(f \otimes g) := \psi_t^*(\{f, g\}) - \{\psi_t^*(f), \psi_t^*(g)\}.$$

Then

$$\begin{aligned}\frac{d}{dt}A_t &= \frac{d}{dt}\left(\psi_t^*(\{f,g\}) - \{\psi_t^*(f),\psi_t^*(g)\}\right)\\ &= -\psi_t^*(\{H_t,\{f,g\}\}) + \{\psi_t^*(\{H_t,f\}),\psi_t^*(g)\}\\ &\qquad\qquad + \{\psi_t^*(f),\psi_t^*(\{H_t,g\})\}\\ &= -\psi_t^*(\{H_t,f\},g\}) + \{\psi_t^*(\{H_t,f\}),\psi_t^*(g)\}\\ &\quad -\psi_t^*(\{f,\{H_t,g\}\}) + \{\psi_t^*(f),\psi_t^*(\{H_t,g\})\}\\ &= -A_t(\{H_t,f\}\otimes g) - A_t(f\otimes\{H_t,g\})\end{aligned}$$

so A_t satisfies the differential equation

$$\frac{d}{dt}A_t = -A_t \circ \left(P_{H_t}\otimes id + id\otimes P_{H_t}\right)$$

with the initial condition $A_0 = 0$. Thus the unique solution is $A_t = A_0 = 0$, so we have that ψ_t^* preserve the Poisson bracket.

Quantification

In order to quantize Poisson automorphisms to obtain morphisms of the associative algebra[2] $(\mathbb{R}[x,y][[\lambda]],\star)$, we replace $P_{H_t} = \{H_t, \ \}$ in equation (7.2.15) by $Q_{H_t} : f \mapsto \frac{1}{\lambda}(H_t\star f - f\star H_t)$.

As in (7.2.12), solutions of

$$\begin{cases}\dfrac{d}{dt}\alpha_t = -\alpha_t\circ Q_{H_t}\\ \quad\alpha_0 = id\end{cases} \tag{7.2.16}$$

write as formal power series

$$\alpha_t = id + \sum_{n=1}^{\infty}(-1)^n\int_0^t\left(\int_0^{t_1}\cdots\left(\int_0^{t_{n-1}} Q_{H_{t_n}}dt_n\right)\circ Q_{H_{t_{n-1}}}dt_{n-1}\circ\cdots\right)\circ Q_{H_{t_1}}dt_1.$$

To show that they are morphisms of $(\mathbb{R}[x,y][[\lambda]],\star)$ *i.e.* they preserve the product $\star$, set

$$B_t(f\otimes g) := \alpha_t(f\star g) - \alpha_t(f)\star\alpha_t(g).$$

Then

$$\begin{aligned}\frac{d}{dt}B_t &= \frac{d}{dt}\left(\alpha_t(f\star g) - \alpha_t(f)\star\alpha_t(g)\right)\\ &= -\alpha_t\circ Q_{H_t}(f\star g) + \left(\alpha_t\circ Q_{H_t}(f)\right)\star\alpha_t(g) + \alpha_t(f)\star\left(\alpha_t\circ Q_{H_t}(g)\right)\\ &= -B_t(Q_{H_t}(f)\otimes g) - B_t(f\otimes Q_{H_t}(g))\end{aligned}$$

[2]here ⋆ denote the Moyal-Weyl ⋆-product or its equivalent simpler form

so B_t satisfies the differential equation

$$\frac{d}{dt}B_t = -B_t \circ \left(Q_{H_t} \otimes id + id \otimes Q_{H_t}\right)$$

with the initial condition $B_0 = 0$. Thus the unique solution is $B_t = B_0 = 0$, showing that α_t is a morphism of $(\mathbb{R}[x,y][[\lambda]],\star)$.

In particular, if we take only one triangular morphism $\phi_t(x,y) = (x, y + tp(x))$ with $p \in \mathbb{R}[x]$ of degree d and $H(x,y) = -\int p(x)dx$ the corresponding hamiltonian, $Q_{H_t} = Q_H$ doesn't depend on t.

Since H is a function of x only, for the Moyal-Weyl $\star$-product (7.2.2), we have

$$Q_H(f) = \frac{1}{\lambda}(H\star_{MW}f - f\star_{MW}H) = -\sum_{k=0}^{\left\lceil\frac{d+1}{2}\right\rceil} \frac{1}{(2k+1)!}\left(\frac{\lambda}{2}\right)^{2k} p^{(2k)}(x)\partial_y^{2k+1}(f),$$

and for the $\star$-product (7.2.3),

$$Q_H(f) = \frac{1}{\lambda}(H \star f - f \star H) = -\sum_{n=1}^{d+2} \frac{\lambda^n}{n!} p^{(n-1)}(x)\partial_y^n(f).$$

The operator Q_H doesn't depend on t, so the differential equation (7.2.16) integrates in

$$\alpha_t = e^{-tQ_H},$$

which is a finite sum when evaluated on a polynomial f, because the compositions Q_H^n only involve successive partial derivations with respect to y of f.

Therefore, we obtain other examples of morphisms α_t of $(\mathbb{R}[x,y][[\lambda]],\star)$ than those of Proposition 7.2.4, and they can be used to obtain a Hom-associative algebra $(\mathbb{R}[x,y][[\lambda]],\star_{\alpha_t},\alpha)$ by the twisting principle.

A

Computations with *Mathematica*

Contents

We give here *Mathematica* commands which were used to carry out various computations to obtain Hom-Lie algebras.

A.1 Computation of the morphisms of $\mathfrak{sl}(2)$

Let sl(2) be the Lie algebra generated by $\{e, f, h\}$ with the bracket given by $[h,e] = 2e, [h, f] = -2f, [e, f] = h$.
We search the morphisms α : sl(2) $\to$ sl(2) satisfying $\alpha([x,y]) = [\alpha(x), \alpha(y)]$. We write $\alpha = (a_{ij})$ in the basis $\{e, f, h\}$. This method was used in [Yau11].

We define the Lie bracket so it satisfies bilinearity properties.

```
Croch[f_ + g_, h_]:=Croch[f, h] + Croch[g, h]
Croch[f_, g_ + h_]:=Croch[f, g] + Croch[f, h]
Croch[t_Real f_, g_]:=tCroch[f, g]
Croch[t_Rational f_, g_]:=tCroch[f, g]
Croch[t_Integer f_, g_]:=tCroch[f, g]
Croch[f_, t_Real g_]:=tCroch[f, g]
Croch[f_, t_Rational g_]:=tCroch[f, g]
Croch[f_, t_Integer g_]:=tCroch[f, g]
{Croch[a[i_, j_]f_, g_]:=a[i, j]Croch[f, g], Croch[f_, a[i_, j_]g_]:=a[i, j]Croch[f, g]}
{Null, Null}
```

We give the values of the bracket on the basis.

```
{Croch[e, e] = 0, Croch[e, f] = h, Croch[e, h] = −2e,
Croch[f, e] = −h, Croch[f, f] = 0, Croch[f, h] = 2f,
Croch[h, e] = 2e, Croch[h, f] = −2f, Croch[h, h] = 0}
{0, h, −2e, −h, 0, 2f, 2e, −2f, 0}
```

We define linearity and multiplicativity properties for the morphism α.

```
alpha[0]:=0; alpha[1]:=1;
alpha[x_ + y_]:=alpha[x] + alpha[y]
alpha[x_y_]:=alpha[x]alpha[y]
alpha[x_^n_Integer]:=alpha[x]^n
alpha[t_Real x_]:=talpha[x]
alpha[t_Rational x_]:=talpha[x]
alpha[t_Integer x_]:=talpha[x]
```

We define the morphism α by a matrix.

```
A = Array[a, {3, 3}]
{{a[1, 1], a[1, 2], a[1, 3]}, {a[2, 1], a[2, 2], a[2, 3]}, {a[3, 1], a[3, 2], a[3, 3]}}
MatrixForm[A]
```

$$\begin{pmatrix} a[1,1] & a[1,2] & a[1,3] \\ a[2,1] & a[2,2] & a[2,3] \\ a[3,1] & a[3,2] & a[3,3] \end{pmatrix}$$

We compute the images of α on the basis.

```
X = {e, f, h}
```

{e, f, h}

```
For[i = 1, i < 4, i++, alpha[X[[i]]] = (Transpose[A].X)[[i]]]
i=.
{alpha[e], alpha[f], alpha[h]}
```
{ea[1,1]+fa[2,1]+ha[3,1], ea[1,2]+fa[2,2]+ha[3,2], ea[1,3]+fa[2,3]+ha[3,3]}

```
MatrixForm[%]
```
$$\begin{pmatrix} ea[1,1]+fa[2,1]+ha[3,1] \\ ea[1,2]+fa[2,2]+ha[3,2] \\ ea[1,3]+fa[2,3]+ha[3,3] \end{pmatrix}$$

We write the polynomials in e, f, h obtained by computation of $\alpha([x,y]) - [\alpha(x), \alpha(y)]$ on the basis. We only need three of this polynomials: $\alpha([e,f]) - [\alpha(e), \alpha(f)]$, $\alpha([e,h]) - [\alpha(e), \alpha(h)]$, $\alpha([f,h]) - [\alpha(f), \alpha(h)]$.

```
poly = SparseArray[{{i_, j_} → If[i>=j, 0,
Collect[Expand[Croch[alpha[X[[i]]], alpha[X[[j]]]] – alpha[Croch[X[[i]], X[[j]]]]],
{e, f, h}]]}, {3, 3}]
```
SparseArray[< 3 >, {3, 3}]

```
Normal[poly]
```
{{0, e(−a[1,3]+2a[1,2]a[3,1]−2a[1,1]a[3,2])+f(−a[2,3]−2a[2,2]a[3,1]+2a[2,1]a[3,2])+h(−a[1,2]a[2,1]+a[1,1]a[2,2]−a[3,3]), h(−a[1,3]a[2,1]+a[1,1]a[2,3]+2a[3,1])+e(2a[1,1]+2a[1,3]a[3,1]−2a[1,1]a[3,3])+f(2a[2,1]−2a[2,3]a[3,1]+2a[2,1]a[3,3])}, {0, 0, h(−a[1,3]a[2,2]+a[1,2]a[2,3]−2a[3,2])+e(−2a[1,2]+2a[1,3]a[3,2]−2a[1,2]a[3,3])+f(−2a[2,2]−2a[2,3]a[3,2]+2a[2,2]a[3,3])}, {0, 0, 0}}

To cancel these polynomials, we extract their coefficients in e, f, h and solve the equations we get, the unknowns being the coefficients a_{ij} of the morphism α.

```
For[i = 1, i < 4, i++,
  For[j = 1, j < 4, j++,
    If[i < j, coefs[i, j] = CoefficientList[poly[[i, j]], {e, f, h}], 0]
  ]
]
vars = Normal[Flatten[A]]
```
{a[1,1], a[1,2], a[1,3], a[2,1], a[2,2], a[2,3], a[3,1], a[3,2], a[3,3]}

```
eqs = DeleteCases[Flatten[{coefs[1, 2], coefs[1, 3], coefs[2, 3]}], 0]
```
{−a[1,2]a[2,1]+a[1,1]a[2,2]−a[3,3], −a[2,3]−2a[2,2]a[3,1]+2a[2,1]a[3,2], −a[1,3]+2a[1,2]a[3,1]−2a[1,1]a[3,2], −a[1,3]a[2,1]+a[1,1]a[2,3]+2a[3,1], 2a[2,1]−2a[2,3]a[3,1]+2a[2,1]a[3,3], 2a[1,1]+2a[1,3]a[3,1]−2a[1,1]a[3,3], −a[1,3]a[2,2]+a[1,2]a[2,3]−2a[3,2], −2a[2,2]−2a[2,3]a[3,2]+2a[2,2]a[3,3], −2a[1,2]+2a[1,3]a[3,2]−2a[1,2]a[3,3]}

```
zeros = Table[0, {i, Length[eqs]}]
```
{0, 0, 0, 0, 0, 0, 0, 0, 0}

```
sols = Solve[eqs == zeros, vars]
```

```
Solve::svars:Equations may not give solutions for all "solve" variables.〉〉
```

$$\Big\{\Big\{a[1,1]\to -\tfrac{a[3,1]^2}{a[2,1]}, a[1,3]\to \tfrac{2a[3,1]}{a[2,1]}, a[1,2]\to \tfrac{1}{a[2,1]}, a[2,3]\to 0\,,$$
$$a[3,3]\to -1, a[2,2]\to 0, a[3,2]\to 0\},$$
$$\Big\{a[1,1]\to \tfrac{1}{a[2,2]}, a[1,3]\to 0, a[1,2]\to 0, a[2,1]\to -\tfrac{a[2,3]^2}{4a[2,2]}\,,$$
$$a[3,1]\to -\tfrac{a[2,3]}{2a[2,2]}, a[3,3]\to 1, a[3,2]\to 0\Big\},$$
$$\Big\{a[1,1]\to \tfrac{1+2a[3,3]+a[3,3]^2}{4a[2,2]}, a[1,3]\to \tfrac{-a[3,2]-a[3,2]a[3,3]}{a[2,2]}, a[1,2]\to -\tfrac{a[3,2]^2}{a[2,2]}\,,$$
$$a[2,1]\to -\tfrac{a[2,2](1-2a[3,3]+a[3,3]^2)}{4a[3,2]^2}, a[3,1]\to \tfrac{1-a[3,3]^2}{4a[3,2]}, a[2,3]\to \tfrac{a[2,2](-1+a[3,3])}{a[3,2]}\Big\},$$
$$\{a[1,1]\to 0, a[1,3]\to 0, a[1,2]\to 0, a[2,1]\to 0, a[3,1]\to 0,$$
$$a[2,3]\to 0, a[3,3]\to 0, a[2,2]\to 0, a[3,2]\to 0\}\}$$

The fitting morphisms α are of the following form.

MatrixForm[A]/.sols

$$\left\{\begin{pmatrix} -\frac{a[3,1]^2}{a[2,1]} & \frac{1}{a[2,1]} & \frac{2a[3,1]}{a[2,1]} \\ a[2,1] & 0 & 0 \\ a[3,1] & 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{a[2,2]} & 0 & 0 \\ -\frac{a[2,3]^2}{4a[2,2]} & a[2,2] & a[2,3] \\ -\frac{a[2,3]}{2a[2,2]} & 0 & 1 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} \frac{1+2a[3,3]+a[3,3]^2}{4a[2,2]} & -\frac{a[3,2]^2}{a[2,2]} & \frac{-a[3,2]-a[3,2]a[3,3]}{a[2,2]} \\ -\frac{a[2,2](1-2a[3,3]+a[3,3]^2)}{4a[3,2]^2} & a[2,2] & \frac{a[2,2](-1+a[3,3])}{a[3,2]} \\ \frac{1-a[3,3]^2}{4a[3,2]} & a[3,2] & a[3,3] \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

These morphisms indeed are invertible since they have non-vanishing determinant, moreover, this determinant is 1 (except for the zero morphism).

Simplify[Det[A]/.sols]

$\{1,1,1,0\}$

We rename the parameters.

Part[MatrixForm[A]/.sols, 1]/.{{1/a[2,1] → λ, a[2,1] → 1/λ, a[3,1] → μ}}

$$\left\{\begin{pmatrix} -\lambda\mu^2 & \lambda & 2\lambda\mu \\ \frac{1}{\lambda} & 0 & 0 \\ \mu & 0 & -1 \end{pmatrix}\right\}$$

Part[MatrixForm[A]/.sols, 2]/.{{1/a[2,2] → λ, a[2,2] → 1/λ, a[2,3] → μ}}

$$\left\{\begin{pmatrix} \lambda & 0 & 0 \\ -\frac{\lambda\mu^2}{4} & \frac{1}{\lambda} & \mu \\ -\frac{\lambda\mu}{2} & 0 & 1 \end{pmatrix}\right\}$$

Part[MatrixForm[A]/.sols, 3]/.{{1/a[2,2] → λ, a[2,2] → 1/λ, a[3,3] → μ, a[3,2] → ν}}

$$\left\{\begin{pmatrix} \frac{1}{4}\lambda\left(1+2\mu+\mu^2\right) & -\lambda\nu^2 & \lambda(-\nu-\mu\nu) \\ -\frac{1-2\mu+\mu^2}{4\lambda\nu^2} & \frac{1}{\lambda} & \frac{-1+\mu}{\lambda\nu} \\ \frac{1-\mu^2}{4\nu} & \nu & \mu \end{pmatrix}\right\}$$

These matrices correspond to automorphisms $\mathfrak{sl}(2) \to \mathfrak{sl}(2)$, $X \mapsto A^{-1}.X.A$ with

$A = \begin{pmatrix} 0 & b \\ -1/b & d \end{pmatrix}$ with $\lambda = -c^2,\ \mu = 2bd$ for the first one,

$A = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$ with $\lambda = d^2,\ \mu = -ab$ for the second one,

$A = \begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}$ with $\lambda = 1/a^2,\ \mu = 1+2bc,\ \nu = -2ac$ for the third one.

A.2 Hom-Lie structures associated to the Jackson $\mathfrak{sl}(2)$ bracket

We obtain the Lie Jackson sl(2) algebra by deformation. It is generated by $\{e, f, h\}$ with bracket given by $[h,e] = 2e, [h,f] = -2(1+t)f, [e,f] = \left(1+\frac{t}{2}\right)h$.

We define this new bracket as before so that it satisfies bilinearity properties.

Croch[f_ + g_, h_]:=Croch[*f*, *h*] + Croch[*g*, *h*]
Croch[f_, g_ + h_]:=Croch[*f*, *g*] + Croch[*f*, *h*]
Croch[t_Real f_, g_]:=*t*Croch[*f*, *g*]
Croch[t_Rational f_, g_]:=*t*Croch[*f*, *g*]
Croch[t_Integer f_, g_]:=*t*Croch[*f*, *g*]
Croch[f_, t_Real g_]:=*t*Croch[*f*, *g*]
Croch[f_, t_Rational g_]:=*t*Croch[*f*, *g*]
Croch[f_, t_Integer g_]:=*t*Croch[*f*, *g*]
{Croch[*a*[i_, j_]f_, g_]:=*a*[*i*, *j*]Croch[*f*, *g*], Croch[f_, *a*[i_, j_]g_]:=*a*[*i*, *j*]Croch[*f*, *g*]}
{Null, Null}
Croch[*t*f_, g_]:=*t*Croch[*f*, *g*]
Croch[f_, *t*g_]:=*t*Croch[*f*, *g*]

We give the values of the bracket on the basis.

{Croch[*e*, *e*] = 0, Croch[*e*, *f*] = (1 + *t*/2)*h*, Croch[*e*, *h*] = −2*e*,
Croch[*f*, *e*] = −(1 + *t*/2)*h*, Croch[*f*, *f*] = 0, Croch[*f*, *h*] = 2(1 + *t*)*f*,
Croch[*h*, *e*] = 2*e*, Croch[*h*, *f*] = −2(1 + *t*)*f*, Croch[*h*, *h*] = 0}
$\left\{0, h\left(1+\frac{t}{2}\right), -2e, h\left(-1-\frac{t}{2}\right), 0, 2f(1+t), 2e, -2f(1+t), 0\right\}$

We search linear maps α which satisfies Hom-Jacobi identity for this bracket. We define the linearity properties for α.

alpha[0]:=0; alpha[1]:=1;
alpha[x_ + y_]:=alpha[*x*] + alpha[*y*]
alpha[t_Real x_]:=*t*alpha[*x*]
alpha[t_Rational x_]:=*t*alpha[*x*]
alpha[t_Integer x_]:=*t*alpha[*x*]

alpha[tf_]:=$t$alpha[f]

We define the map α by a matrix.

A = Array[a, {3, 3}]

{{a[1, 1], a[1, 2], a[1, 3]}, {a[2, 1], a[2, 2], a[2, 3]}, {a[3, 1], a[3, 2], a[3, 3]}}

MatrixForm[A]

$$\begin{pmatrix} a[1,1] & a[1,2] & a[1,3] \\ a[2,1] & a[2,2] & a[2,3] \\ a[3,1] & a[3,2] & a[3,3] \end{pmatrix}$$

X = {e, f, h}

{e, f, h}

For[i = 1, i < 4, i++, alpha[X[[i]]] = (Transpose[A].X)[[i]]]

i=.

{alpha[e], alpha[f], alpha[h]}

{$ea[1,1]+fa[2,1]+ha[3,1]$, $ea[1,2]+fa[2,2]+ha[3,2]$, $ea[1,3]+fa[2,3]+ha[3,3]$}

MatrixForm[%]

$$\begin{pmatrix} ea[1,1]+fa[2,1]+ha[3,1] \\ ea[1,2]+fa[2,2]+ha[3,2] \\ ea[1,3]+fa[2,3]+ha[3,3] \end{pmatrix}$$

We have to find the coefficients of α such that the Hom-Jacobi identity is satisfied.

Jac = Croch[alpha[e], Croch[f, h]]
+Croch[alpha[f], Croch[h, e]]
+Croch[alpha[h], Croch[e, f]]

$2h\left(-1-\frac{t}{2}\right)a[2,2]+4ea[3,2]+a[1,3]\mathrm{Croch}\left[e,h\left(1+\frac{t}{2}\right)\right]+2a[1,1]\mathrm{Croch}[e,f(1+t)]+a[2,3]\mathrm{Croch}\left[f,h\left(1+\frac{t}{2}\right)\right]+2a[2,1]\mathrm{Croch}[f,f(1+t)]+a[3,3]\mathrm{Croch}\left[h,h\left(1+\frac{t}{2}\right)\right]+2a[3,1]\mathrm{Croch}[h,f(1+t)]$

We extract coefficients of this polynomial in e, f, h and solve the corresponding system to cancel them, unknown being the coefficients a_{ij} of the map α.

Collect[ExpandAll[Jac], {e, f, h}]

$h\left(2a[1,1]+3ta[1,1]+t^2a[1,1]-2a[2,2]-ta[2,2]\right)$
$+f\left(2a[2,3]+3ta[2,3]+t^2a[2,3]-4a[3,1]-8ta[3,1]-4t^2a[3,1]\right)$
$+e(-2a[1,3]-ta[1,3]+4a[3,2])$

eqs = CoefficientList[Collect[ExpandAll[Jac], {e, f, h}], {e, f, h}]

$\{\{\{0, 2a[1,1]+3ta[1,1]+t^2a[1,1]-2a[2,2]-ta[2,2]\}$,
$\{2a[2,3]+3ta[2,3]+t^2a[2,3]-4a[3,1]-8ta[3,1]-4t^2a[3,1], 0\}\}$,
$\{\{-2a[1,3]-ta[1,3]+4a[3,2], 0\}, \{0, 0\}\}\}$

equas = DeleteCases[Flatten[eqs], 0]

$\{2a[1,1]+3ta[1,1]+t^2a[1,1]-2a[2,2]-ta[2,2]$,
$2a[2,3]+3ta[2,3]+t^2a[2,3]-4a[3,1]-8ta[3,1]-4t^2a[3,1]$,
$-2a[1,3]-ta[1,3]+4a[3,2]\}$

vars = Flatten[A]

$\{a[1,1], a[1,2], a[1,3], a[2,1], a[2,2], a[2,3], a[3,1], a[3,2], a[3,3]\}$
sols = Solve[equas == {0, 0, 0}, vars]
```
Solve::svars:Equations may not give solutions for all
 "solve" variables.⟩⟩
```
$\left\{\left\{a[3,2] \to -\frac{1}{4}(-2-t)a[1,3], a[2,2] \to -(-1-t)a[1,1], a[2,3] \to \frac{4(1+t)a[3,1]}{2+t}\right\}\right\}$

We rename the parameters.

**MatrixForm[*A*]/.sols/.{*a*[1, 1] → *a*, *a*[2, 1] → *b*, *a*[3, 1] → *c*,
a[1, 2] → *d*, *a*[1, 3] → *k*, *a*[3, 3] → *l*, *t* → *q* − 1}//Simplify**

$$\left\{\begin{pmatrix} a & d & k \\ b & aq & \frac{4cq}{1+q} \\ c & \frac{1}{4}k(1+q) & l \end{pmatrix}\right\}$$

Specifying coefficients, we recover the twists corresponding to the already known deformations.

**MatrixForm[*A*]/.sols/.{*a*[1, 1] → 1 + *t*, *a*[2, 1] → 0, *a*[3, 1] → 0,
a[1, 2] → 0, *a*[1, 3] → 0, *a*[3, 3] → 1 + *t*}//Simplify**

$$\left\{\begin{pmatrix} 1+t & 0 & 0 \\ 0 & (1+t)^2 & 0 \\ 0 & 0 & 1+t \end{pmatrix}\right\}$$

**MatrixForm[*A*]/.sols/.{*a*[1, 1] → (2 + *t*)/(2(1 + *t*)), *a*[2, 1] → 0, *a*[3, 1] → 0,
a[1, 2] → 0, *a*[1, 3] → 0, *a*[3, 3] → 1}//Simplify**

$$\left\{\begin{pmatrix} \frac{2+t}{2+2t} & 0 & 0 \\ 0 & 1+\frac{t}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

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